Nonus:

The Nine Bead Abacus, its Theory and Derivation

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I. Introduction

It may come as some surprise to find that work is still being done with that ancient calculating device, the abacus. After all, we live in an era when the digital revolution is occurring, real time, right before our very eyes. The means and methods by which we work, play and communicate are changing at an unparalleled rate in human history. Yet, though the gap between the so-called underdeveloped nations and the "first world" nations is diminishing, many millions of people continue to use a version of the abacus in the exchange of everyday commerce, even in the 21^{st} century.

We have witnessed the recent phenomenon of the Japanese Soroban, its development, popularity and refinement during the post-WWII era into a device of high efficiency and methodology. This occurred in spite of Japan having been in the midst of a technological revolution, which, it would seem, would supplant the role of the bead-frame calculator to that of an antiquated, dusty museum piece.

In this paper, we will cover a brief history of the abacus in both ancient and modern eras, and discuss the evolution of the device and refinement of its operation. Then, we will touch on the role that the bead-frame calculator has in modern America, in commerce, at home, and in the classroom. Finally, the author will present the results of research he has conducted on an updated, westernized nine-bead abacus, the *Nonus*, and its operation. We will also discuss the concept of the "minimal abacus", and some practical examples.

II. The Abacus as Practical Tool

First, let us ask the question "why the fascination with the abacus?" For anyone who has had opportunity to manipulate its beads, the abacus induces a curiously satisfying sensation to the operator. But in a more general sense, why has the abacus been developed and used, throughout centuries, by many differing cultures in such widely differing contexts? Its development appears to have been simultaneous and independent throughout differing ages and locales, despite the unifying effects of nation conquering nation and the spread of commerce.

We contend that the explanation occurs on several levels. First, the abacus, as an anthropological tool, appears to be what is termed a *primary tool*. That is, it is on the same level as the inclined plane, the wheel, the block and tackle or the lever. Its simultaneous occurrence in such widely separate venues, and its common sense, easily accessible technology appeal to this view. The utility of this simple, efficient device for satisfying the arithmetical needs of cultures throughout history speaks of its primacy. The author's contention is that, as a primary tool, the abacus is under-appreciated and its role as a cultural icon is less than adequately documented, not only in the context of history, but in the present time even as cultures which, though historically abacus "rich", are themselves undergoing major modernization.

Secondly, the abacus is a *fundamental crystallizer.* That is, it embodies within its design such mathematically fundamental concepts as place value, grouping, sets, one-to-one correspondence, parity and complimentary numbers. As such, it is capable of being an almost ideal interface between the abstract conceptualization of numbers in their purest sense, and their concrete representation to "real" material objects. The manner in which it does so is as a tool ideally suited to *crystallizing* that which comes forth from man's imagination numerically into concrete representations. Through the hand of the craftsman and operator, abstractions are fashioned into a utilitarian functionality, much like the hand to the plow, or the brush to the canvas, or the pen to paper. The widespread use and independent development of the abacus is testimony that people take to the bead-frame calculator in a natural, unassuming manner.

Thirdly, the methodologies developed to perform functions on the abacus allow such operations as addition and subtraction to take place in an almost automatic fashion. It is as if the laying down of numbers (representing them on the abacus) by following several simple rules, causes the answer to "form itself' in a natural fashion without undue mental concentration or strain. The abacus is a venue whereby one can witness the transformation of abstract operating principles into a functioning machine for performing everyday calculations. That is, without those principles of operation, the abacus is no more a calculating device than a sack of pebbles; it would merely remain a frame with beads strung on rods. In the hand of the operator, this object becomes a calculator *because of* the application of principles of operation. As numbers are entered onto the bead frame, the operator is not conscious of "doing calculations" as much as adhering to certain rules. Almost as if by magic, the answer has been formed in the pattern of beads placed in front of the operator, who is conscious not of having performed the actual calculation, but of entering a series of numbers onto the columns of beads using predefined rules. Thus, the act of data entry, and the calculation itself, occurs as a simultaneous step.

Perhaps the magic of the device comes from this sudden realization that the calculation process is imbedded within the mechanism of data entry. Indeed, one gets this impression from watching a room full of students, all silent, yet intent upon the concentrated activity of adding up columns of numbers from a sheet next to them. Each holds their soroban's frame with their left hand in a deft, precise manner, while the thumb and first finger of their right hand perform a flurry of dance-like moves. First right, then with a quick flick of the left wrist followed by a swift, clackety stroke of the right again, the calculation is cleared and the frame made ready for the next series of numbers. All the while, the silence is broken by the quiet, satisfying din of the wooden beads clicking constantly, becoming a kind of background music which one soon learns to ignore, until the exercise is ended by the instructor, at which point the silence in the room becomes deafening. The speed that the student soon displays in calculations is brought about by the application of finely developed rules of bead manipulation. These rules become just as important in developing speed, as do the mental rules of lying on and taking off numbers.

Historically, the Japanese have not always embraced the soroban as an icon symbolic with Japanese culture. After its adoption and culturalization from China in the 14^{th} and 15^{th} centuries, the soroban is seen in woodcuts as being an implement of commerce employed mainly by the overlord class and Samurai in the transactions common with the wealthy. When it adopted its unique, Japanese style of beads, and lost its extra upper bead, we are uncertain. However, it is a classic Japanese trait to borrow from other cultures and soon refine, improve and apply to local needs in such a fashion that it becomes truly theirs. Curiously, the abacus soon went into decline after contact in the mid-to-late 19^{th} century by European and American explorations and military missions. In their typical manner, the Japanese took to western culture just as quickly as previously with the Chinese, until the soroban was relegated to the lower trades as an anonymous tool of commerce. Not until post WWI, and especially after WWII, was the subject of soroban study applied in the country's schools, and embraced as a once-forgotten icon now rediscovered. As the windmill is to the Netherlands, so has the soroban become to Japan. During this time, too, was the soroban to lose its extra lower bead, and the rules of bead manipulation and efficient number placement to be developed to a high level of sophisticated technique such that its study is a subject of higher education even today.

Venturing across the Sea of Japan to the Korean peninsula, one comes into contact with the soroban's cousin, the Chinese suan pan. The very abacus that the Japanese adopted and improved upon centuries ago is still in use today throughout eastern Asia. Indeed, it seems that anywhere Chinese culture is or has been in dominance, we find evidence of the suan pan. Taking a taxi to any one of South Korea's small shops or cottage industries, such as during the author's visit to a clothing factory, reveals several abaci situated on the counter where transactions are made with customers. So common is the abacus in these cultures that operating knowledge is almost universal to anyone who has to deal with cash transactions or the manipulation of numbers on a regular basis. It is indeed ironic to consider that in many of these countries the manufacture and export of cheap, disposable electronic calculators is commonplace, yet the "common" people continue use of the abacus, perhaps as a means of retaining cultural identity.

Today, one finds that in appearance the soroban resembles the Chinese suan pan in certain respects. The suan pan's frame is squarer than the soroban's long, sleek rectangular look. Both types of abacus have a horizontal frame member dividing the rods into a thin, upper region, and a wider, lower region. In the ancient cultures of both China and Japan, these areas are known respectively as heaven and earth. The suan pan possesses two round beads in the upper part of each rod, and five below on each rod. The soroban has one diamond shaped bead in each upper portion, and four below. It should be noted that soroban constructed before WWII usually had five beads in the lower portion; it was only in recent times that the extra bead was determined to be extraneous.

Most examples of the soroban examined by the author are constructed with finer materials and workmanship than that of the suan pan. Some older, larger soroban have tongue-and-groove joinery at the corners, while the suan pan typically has crudely mitered joints. The soroban have flat wooden braces on the rear of the frame, with newer models also having round dowels with metal bushings, pinned in place to allow a secure joint through years of use and wear on the wooden components. A curiously common feature on most old and new soroban models is that the two side frame members have their top surfaces arched, as viewed from the side. Why this feature is incorporated remains a mystery to the author. In both types of oriental abacus, the "heaven" bead in each rod has a value of "5", while each "earth" bead has a value of "1" of that place value, thus representing a *quinary*-based grouping scheme.

As we trek from China through central Asia we find the abacus reappears in a different form - that of the Russian *schyotti*, or Asian ten-bead abacus. Its widespread use throughout Asia, from Russia to Iran, Turkey to Mongolia, in such widely diverse social groups - from primitive herdsmen to sophisticated Moscow shopkeepers bear testimony to its commonplace, almost primal, nature. The typical schyotti is arranged such that the rows of beads are horizontal, rather than vertical as in the oriental versions. The metal or wooden rods that the beads slide on are arched in the middle, so those beads slid from the right to the left are not easily moved back by mistake. Also, one finds as a common feature the two middle beads in each row to be colored differently from the rest, thereby aiding in easily grasping the size of bead groupings at a glance. Typically, the application of the ten-bead abacus is in monetary calculations, so that the number of beads on some of the rods is less than ten, in accordance with the Russian denominations of ruble and kopeck. Some versions are also seen with a "decimal place" row, containing just a single bead, underneath of which are additional rows representing the fractions ¼, ½ and ¾. It has been suggested that, of all forms of the abacus, the ten-bead schyotti is the most commonly used and readily accepted version anywhere in the world, easy to construct and use, and found in application throughout the widest diversity of cultures throughout Asia. This seems to hold true even in the face of the Japanese soroban's more sophisticated development and refinement. Indeed, the schyotti has a striking similarity to the preschool toy bead frame evident in modern America and Europe. There is good reason for this similarity, for, as we shall see, it too can trace its roots back to the Asian ten-bead abacus.

Young and old alike are drawn to the bead frame, perhaps for differing reasons. For the old, it is the reassurance of things familiar from long ago, or perhaps a silent protest against the uncontrollable approach of the future. For the young, regaining a sense of national identity from the pervasive, global clutches of post-modernist western culture fuels a renewed acquaintance with one's classic, cultural icons; a nationalism not as much political as cultural, a returning to the roots. In the case of the abacus, however, we find those roots to be not merely superficial, as in America's current fascination with the culture of the 1950's, for instance, but deep, ancient, even primordial. Its thread through history spans not mere decades, as does America's, but centuries, kingdoms and cultures.

III. Origins

The roots of the counting frame recede into the dim stretches of antiquity. Accurate dates are difficult to place on each of its evolutionary stages. Suffice it to say that humans have applied symbolism to represent numerical quantities since our origins. Whether knotting of ropes, bags of pebbles or lines drawn in the sand, the recording of numbers and actual calculations are evident as far back as the Babylonian culture, thanks in part to the permanence of clay-tablet engravings. The lack of permanent recordings of such activity does not preclude us from assuming that mechanical contrivances of some sort have been used in virtually every culture from prior to Babylon, up to the classical Grecian era.

It is from the Grecian era that we find recorded for posterity details of the abacus in everyday use. Specifically, the *Darius Vase* and *Salamis Tablet* are the best-known examples preserved. On the side of the Darius Vase we find painted a scene of commerce whereby a treasurer is busy at his table, with noblemen and merchants surrounding him. Depicted on the face of the table are numbered rows, with tokens or counters placed so as to represent the quantity 1731 4/6. In his left hand, the treasurer holds a clay tablet; upon which is recorded the computations from the counting board. The Salamis Tablet is an actual example of the type of counting board depicted on the Vase. It is worthwhile noting that the word “abacus” is derived from Latin, and earlier from Greek, to mean a flat surface or table. The term is used in architecture when referring to the flat plate at the top of a Doric column.

The basic layout of the Grecian abacus is a series of parallel lines arranged in columns upon a board, tablet or table. Inscribed at the head of each column is a symbol representing the place value for that position, with the smallest denomination starting at the right and proceeding to the left in order of increasing value. With individual counters, or *calculi*, placed in the space between two lines, we find the numbers represented in a quinary fashion. When five counters occupy one column, they are removed, and represented by a single counter placed on the line to the immediate left. Numbers between five and nine are represented by a combination of the two; that is, to represent a quantity of eight, the column would contain three counters, and the line to the left would contain one counter, symbolizing an additional five.

The process of addition was straightforward, containing two basic steps. First, both numbers to be added together were recorded on the abacus, one below the other, with some degree of space separating them vertically. Then, the two numbers were grouped together, and the resulting quantity was reduced to a readable answer by the quinary grouping method: if five counters were found in any one column, they were removed and replaced by a single counter on the *line* to the immediate left. If two counters were found on a line, they were removed and replaced by a single counter on the *column* to the immediate left. This process was continued, usually from right-to-left, until a readable answer resulted; that is, one in which no column contained more than four and no line contained more than one. The answer was read straight off as a quantity indicated by place values, much like the later Arabic notation.

Here, the similarity in quantity to Roman numerals is evident, in regards to the quinary groupings of five, fifty, five hundred, etc. Where the abacus differs with Roman numerals is in the context of place value notation. Calculations with straight Roman numeral notation were practiced at times, but the process lacked place values. As such, the obvious advantage of the abacus came to the fore. It is obvious, too, that the Greeks, and later the Romans, practiced a sophisticated algebra, having an understanding of the place value concept not through the application of written notation, but through usage of the abacus. As indicated on the Darius Vase, monetary calculations were performed on the counting board, and the results recorded in hand-written notation either in clay, papyrus or paper. This practice was so useful, especially later, in semi-illiterate Europe that it continued throughout the middle ages and well into post-renaissance Europe in private homes, merchant shops and various banking houses.

Somewhere between the Greek and Roman eras, a portable hand-abacus came to be developed. We see examples of this in several European museums in the form of a bronze hand abacus using spheres of clay or bronze, which rolled in grooves engraved into the metal plate. Obvious of Roman origin, the grooves are organized in upper, short rows and longer, lower rows, and utilizing quinary grouping very similar to the oriental abacus of today. It is conjectured that perhaps the Chinese abacus is of Roman origin. Indeed, it has been well documented that the trade routes of the Roman Empire stretched as far as China. The Roman culture, however, also continued usage of the Grecian counting board style of abacus.

Though the Roman Empire slowly declined through the first few centuries of the new millennium, Roman culture continued to dominate European history for the next thousand years. Latin was retained as the “official” language of, not only the church, but also the educated, and especially the monastic. It is not surprising that here we would find preserved the counting board, or reckoning table, with it’s *calculi*, or counters. For even the terminology of the Romans is preserved in the usage of the term *purgatio rationis*, or cleansing of the calculation, for the process of quinary grouping which was the final step of reduction in computation with the counting board.

Throughout the early middle Ages, we find the abacus used primarily in the monastic culture. Not only were addition and subtraction commonplace, but methods for multiplication and "iron division" have been documented as well. From the 13th century and onward the abacus became a tool for the common person and merchant, being found in the houses of the noble and ignoble alike. Perhaps the best illustration of the pervasiveness of the abacus in western culture during the later middle Ages is in the terminology that has been handed down to us. We have the terms check, counter, checker, checkerboard, chess, calculate, bureau, burlesque, divan and bank among others, which are directly descended from the culture of the counting board.

Since the 13th century, the orientation of the reckoning board was such that the place values were in horizontal rows, rather than the vertical columns of the Roman abacus. By the 16th century this style of abacus had long been in existence, as indicated by numerous illustrations from the period. It is interesting to conjecture that perhaps the Russian schyotti, with its horizontal rows of beads, is derived from the abacus of the late middle Ages. Perhaps central Asia, stretching from Western Europe to the Orient, merged the logic of the counting board with the bamboo-inspired construction of the Chinese suan pan. Whatever the case, the European counting board was found in two versions: the "line board" for the manipulation of abstract, unitless numbers, and the "coin board", whose rows were labeled in denominations of the currency in use, which may not necessarily have been of a base ten number system. We can imagine a coin board built for modern currency, with rows for cents, nickels, dimes, quarters, etc. Of course, its application would be impractical because of the decimal notation with which we record financial figures. In our case, a line board would be more applicable.

Another variation of the European counting board is the so-called reckoning cloth, whose lines and operation mimicked that of its fixed brother. These were used no doubt for purposes requiring portability, such as an official whose duty it was to travel from province to province and verify the financial dealings with the local magistrates. Several Bavarian examples of these exist, and date from about the 17th century. The cloth was laid out on a conventional table or flat surface, or even the ground, with a sack of *calculi*, or *jetons*, at hand for casting.

Yet another variation of the coin board dates from 15th century France, where the horizontal lines which defined the place values were no longer drawn out. The place values were instead indicated on the left side of a blank table by the placement of a "tree", or column of coins, which marked the values. The tokens or jetons were then cast in rows to the right of the value-indicating coin.

So pervasive was the counting board in European culture that we find numerous references to it in everyday speech, as well as the sayings of Martin Luther and the plays of Shakespeare. "Thus the Jews placed the counters on the lines and reckoned how many Canaanites there were and what a small number of Israelites there were", quotes Luther. In *Troilus and Cressida*, Shakespeare asks, "Will you with counters sum the past proportion of his infinite?" And again Luther: "The Devil is in a rage and he throws the hundred into the thousand, and thereby creates so much confusion that no one knows what to think." This last saying is used even in Germany to this day. Martin Luther often used references to the counting board in his sermons and daily speech. It is said of Goethe "No toy fascinated him more than his father's counting board, on which he would reproduce the constellations of the stars with counters." In *Faust* we read, "Did you think they'd give you real money and goods? In this game even worthless counters are far too good for you."

During the invasion of Russia in 1812, Napoleon had an engineer by the name of Poncelet, who later founded Projective Geometry. Poncelet, upon capture by the Russians, was brought to Saratov, on the Volga. During his stay there, he noticed the peasantry and their use of the schyotti, the Russian ten-bead abacus. So impressed was he that, upon returning to France, he introduced what came to be called the *boullier* into the schools of Metz. From there, it spread all over France and Germany and into America, where we find it today, relegated to the ranks of childhood play toy.

Numismatists will appreciate the parallel history of the casting counter, also called the *jeton* in France, and *calculi* from ancient Rome. Some examples date back from before the roman era. These differed from coinage principally because of their having been minted on only one side, and were usually not cast from precious metals such as silver or gold. Many of these counters were inscribed with the insignia of the banking house or royalty upon whose tables they were cast. So unaware are many today about the origin and use of the casting counter in abacus calculations, that many numismatic experts are baffled to explain the purpose of one-sided minted “coins”, found in museums such as the National Numismatic Museum in Colorado Springs.

Across history and geography we find stretched the story of the abacus, in its two parallel forms, the counting board and the bead frame. So prevalent has its presence been that our very language owes its roots, in part, to the terminology of the counting board and the counters cast thereon. The question therefore becomes obvious: if the abacus has had such far reaching influence in the past, does its utility at the present moment apply only to certain cultures respective of past traditions, such as China or Japan? Or, is it relevant to contemporary western culture as well, aside from the baby crib and playpen?

IV. The Once and Future Abacus

Let us imagine it to be late in the 1980’s, in a mid-sized southwestern city in the United States. Albuquerque is a town with a diverse business climate, flanked by the traditions of the past and the high technology of the post-nuclear era. Included in this mix is a plethora of small businesses, including electronic repair shops. In one of these consumer electronics service centers is a service technician, intent upon totaling a bill in order to notify a customer of a repair estimate. The numbers are not added up using grade school hand addition on paper, or a pocket calculator, as do the other technicians. Instead, this “ticket” is being totaled using an obviously homemade bead frame abacus. This particular abacus frame is made of stained pine, with the rods a stiff, green-coated metal wire, of the type used in floral arrangements. The black and white, round faceted beads are of the type found in craft stores. Though crude in appearance, the prototype nine-bead abacus displays a high degree of functionality, as is evidenced when the tax rate is calculated in order to finish the estimate. It is humorous to watch the reaction from fellow employees, as well as the business owner, when each ticket is added up, and the estimates called in, with no errors. Something is oddly amiss here when the technician drops his oscilloscope probe and reaches for his bead frame calculator.

As is apparent, the scene described above is not fictitious, but from the author’s personal experience. Answering the question of the relevance of the abacus in contemporary America is no more difficult than looking into the homes and workplaces of ordinary people and seeing how prevalent is the electronic pocket calculator. Imported from countries that are traditionally abacus rich, these have become objects of everyday life since the early 1970’s. Oddly, they have no predecessor in American culture, other than hand calculation on paper, since the slide rule had been primarily a tool of science and engineering, not everyday use. It was not as if there had been an abacus native to North America, made obsolete by the emerging electronics revolution. Rather, the pocket calculator satisfied a need, which lie dormant within the milieu of modern life. The author’s contention is that, primarily for addition and subtraction, the abacus has a valid role to play in the affairs of contemporary culture in America.

Certainly, the soroban plays such a role in modern, high tech Japan. Constructed from quality hard woods by craftsmen working in traditional cottage industries, the soroban is used, not only for the teaching of mathematics in school, but by tradesmen, merchants and everyday people. Pride is seen in the ability to operate the soroban with high efficiency, which Japan encourages by incorporating a ranking system with a high degree of competitiveness, through periodic testing and regular contests. One can also log onto the World Wide Web and visit the home page of several soroban manufacturers. These include a virtual soroban museum with a Java application that produces an interactive, virtual abacus for performing actual calculations. The center of the high-tech, consumer electronics revolution, Japan has not shown disdain for the ancient bead frame calculator, but has embraced it as an icon of its cultural history, while improving upon it.

The author first became aware of the abacus while in high school, when he purchased a soroban from a local Japanese import shop, and learned its use with the aid of an English language book written by a Japanese soroban expert. Studying the soroban, one begins to understand that the primary mental difficulty to be overcome is the concept of *nested complimentary steps*.

Here, the concept of complimentarity needs to be explained, along with the basic methods of addition and subtraction. The soroban uses two types of compliments, five's and tens. As an example of five's compliments, consider that the compliment of four is one, the compliment of three is two, etc. The five's compliment of a number is the amount, that when added to the number, totals five. In the same manner, an example of tens compliments is that eight is the compliment of two; four is the compliment of six, etc. Performing a complimentary step involves performing the *opposite* operation onto the *compliment* of the original number.

An example of an actual calculation on the soroban using five's compliments is the addition of four plus three. First, four beads are pushed up toward the dividing bar. It should be first noted that the abacus is "cleared" by momentarily tilting the abacus up to allow all the beads to slide down their rods, and then to run one's forefinger along the top edge of the dividing bar, to push the upper beads away from the bar. Thus, the abacus registers zero, and is ready for use. Now, we have already entered the number four onto one of the rods of our bead frame. Incidentally, it matters little at this point which rod that we use. Usually, the right-hand-most rod is assumed to be the unit's column. However, the soroban has points marking every third column, for the placement of the decimal point when entering multiple numbers, such as in multiplication or division, and also to indicate the commas used to denote the thousands. Now, we are ready to add the three to our already entered four.

Here we will come to the sudden realization that there are no more beads in the lower part of our column to be employed in our calculation. At this point the novice may be dismayed, thinking that there is a level of complexity out of scale with the simplicity with which this operation can be performed using pencil and paper. The point of the operation is that the same principles, which come into play in simple problems, are used over and over again in more complex ones. There is no fundamental difference in complexity between adding two one-digit numbers, and adding a column of fifteen-digit numbers. It's just a matter of scale.

Now, back to our problem. Instead of wondering where the other three beads are going to come from in order to complete the operation, we instead focus on the concept of the five's compliment. The rule is simple: if the addend cannot be entered directly, then you *subtract* the five's compliment, and add an additional five-bead. Thus, in our problem, we need not worry. The five's compliment of three is two; we *subtract* two from the four already entered, and move down a five bead. The answer forms itself automatically as seven, in the form of one five-bead, and two one-beads. This concept works equally well when dealing with any five's compliment problem.

We may also consider a ten's compliment problem. Consider eight plus eight. Here, we enter eight by moving down a five- bead, and three single-value beads are moved up. It should be noted that, according to the rules of bead manipulation developed in Japan, these two operations could be performed simultaneously in a single, pinching-like motion of the forefinger and thumb. Now, we note that eight more cannot be directly entered onto the same column of the frame. So, we employ the use of the ten's compliment by *subtracting* two, which is the compliment of eight, and entering an additional bead onto the next column to the left, representing ten. The physical rules of bead manipulation require us to do the last two steps simultaneously, by subtracting the two from the one's column with our index finger, while the thumb pushes up the single ten-bead, all in a continuous twisting motion of the right hand. The answer forms itself naturally as sixteen, represented as a one-bead in the ten's column, and a five-bead and a one-bead on the first column.

Even without an actual abacus on which to perform these example calculations, the reader is encouraged to use a handful of pennies, in place of beads. Draw a series of six-inch long parallel vertical lines on a sheet of paper, each about an inch apart, then a horizontal line through all of them, about a quarter of the way down from the top. These are your rods. Then, place the pennies to represent a "cleared" abacus, with the single upper beads all the way up, and the four lower beads in each column all the way down. The horizontal line forms the “bar” toward which counters are slid to represent values on each rod.

Subtraction is performed on the abacus with equal ease as addition. An example of a simple problem is eight minus three. First, eight is entered onto the first column. Then, three one’s beads are removed, completing the problem with an answer of five. An example of a complimentary problem is sixteen minus seven. First, using a one-bead in the ten’s column, and a five and a one-bead on the first column, sixteen is entered. Since we do not have seven to subtract from the first column, we perform the compliment by *adding* three, the compliment of seven, and *subtracting* a single ten-value bead from the second column, yielding us an answer of nine. Thus, subtraction operations on the abacus are just the inverse of addition; the concepts remain the same.

To summarize, whenever a compliment is used in *addition*, a single bead of the next higher value is *added*; in *subtraction*, a single bead of the next higher value is *subtracted*. And in both addition and subtraction, using the opposite operation from the original problem completes the complimentary step. That is, in addition, the compliment is subtracted, while in subtraction, the compliment is added. To the reader, this technique may appear to be simply another way to view the process we called borrowing and carrying, which we learned in grade school, and is now termed “regrouping.” Such is not the case, as the technique of hand calculation on paper requires us to memorize the addition tables beyond nine plus nine, in the case of adding a column of figures, as well as memorize a running total. The abacus method requires us to only know the compliment of a number, and the appropriate rule to apply based to the operation in question. One is thus not adding or subtracting numbers in one’s head, as on paper, but letting the abacus method do the work for us.

The concept of five’s and ten’s compliments is fairly straightforward to grasp; with a little practice the reader could be very adept at addition and subtraction if that is all there was to it. However, we may also experience problems involving *nested compliments*, such as six plus seven. Let’s look at this more closely. First, a single five and a single one are entered onto the right hand column of the soroban, representing a six. Then, we note that we cannot directly add seven more; but we remember that the ten’s compliment of seven is three, so we should be able to subtract the three and add a single bead on the ten’s column, in order to finish the problem. It can immediately be seen that we cannot directly subtract three from the six, for we only have a single one-value bead available. So, we must perform a five’s compliment step in order to continue the operation. Thus, we *add* two, which is the five’s compliment of three, and subtract the five-bead. Then are we able to add the ten-bead on the next column to the left, which completes the ten’s compliment step. Thus, we are left with the answer thirteen, represented by a one-bead in the tens column and three one-beads in the right hand column. This problem is an example of a five’s compliment step *nested within* a ten’s compliment operation. It is problems like these that cause the modern soroban to be looked upon by the westerner as a device requiring much mechanical dexterity and mental gymnastics in order to perform efficiently.

V. Further Developments

The realization of nested complimentary steps caused the author to look to the Asian ten-bead abacus as an alternative. While the experienced are adept at soroban operation due to much practice, the average person who may pick up a pocket calculator may not be so inclined as to invest the time required to master basic operations, such as those described here, on the abacus. It became immediately obvious that five’s compliment steps do not exist on the ten-bead abacus, such as the Russian schyotti, *since it is not quinary based*. Here, one need only memorize the ten’s compliments table, and operations of any length can be performed with minimal difficulty. This is evident when reviewing the problem above, as we find that adding seven to six on the schyotti is no more taxing than subtracting three, which is the ten’s compliment of seven, and adding the single ten-bead.

After reviewing the history of the soroban in Japan, and the refinements that took place since adoption of the Chinese suan pan, the author considered that the ten-bead abacus could be further modernized in a similar fashion by removing the extra bead on each row. To further clarify this, consider the Arab number system that we have adopted. Each place value in our base-ten system can represent nine different values, symbolized by the characters one through nine, as well as zero. Since zero on the abacus represents no beads entered on a column, we only need to represent values from zero to nine on each of the abacus' rods. This is the innovation that the Japanese made with the quinary abacus, in removing the second upper bead, and fifth lower bead from the Chinese suan pan (which originally could represent monetary values up to fifteen in each column). In the case of the ten-bead abacus, nine beads are all that are required to represent any decimal-based number in a column, the tenth bead being superfluous.

The second refinement made by the author was in the orientation of the abacus. You may note that the Russian schyotti is placed such that the rods are horizontal, with the numbers represented vertically in ascending order of place value. It seems more in keeping with the written order of numbers to hold the nine bead abacus flat, in a fashion similar to the oriental abacus, so that the rods are vertical, with the place values in ascending order from right to left, as in our written place value system. It is felt that this representation of numbers is more in keeping with the manner in which we mentally cognate, and visualize, numerical values.

Thirdly, the problem of visual recognition was addressed. It may be recalled that the schyotti utilizes two middle beads that are colored differently from the others in that row, in order to facilitate the visual grasping of number groupings. After experimenting with many variations of colored beads, it was found that an alternating pattern of light and dark beads was the most efficient. As well as facilitating easy recognition of number groups, an additional advantage to this configuration is that the odd or even parity of a group of beads is revealed in the colors of the end beads. For instance, a group of five beads can be easily distinguished because *odd* groupings have *identical* end colors. Six is easily distinguished from five because, although similar in length, a group of six has *opposite* colored ends, being of *even* parity. This principle of end coloration verses grouping parity helps to overcome, in large measure, the disadvantage the nine-bead abacus may appear to have when compared to the soroban, which is in regards to the multitude of beads on each row.

Subsequent construction of basic models, and numerous operations that compared the nine-bead abacus against the soroban revealed the ease with which numbers can be added and subtracted. In fact, with the memorization of the ten's compliments table, abacus operation can become second nature, virtually intuitive, with the novice operator coming up to speed much sooner on the nine-bead abacus, as nested complimentary problems need not be addressed.

Since the nine-bead abacus dovetails so nicely into the concepts of number theory, place value, finger counting and parity, and represents a physical analogy to our base-ten number system, it would seem natural that its application to education would become commonplace. The author's contention is that the grade school experience in arithmetic can be greatly enhanced by the inclusion of the nine-bead abacus as a teaching aid in the classroom. The advantages do not end with the lower grades; competitive, performance-oriented contests and ranking systems can be developed, in the spirit of the Japanese soroban, such that older students would be compelled to continue study and practice with the bead frame well into adulthood. Indeed, adults could be taught directly the nine-bead abacus as an entirely new subject, in courses of adult remedial education, or in the case of those seeking personal improvement via continuing education, non-credit college courses.

Also, there is little reason why the nine-bead abacus could not supplant the electronic pocket calculator for applications of addition and subtraction, even outside of the classroom. Organizations could be formed, whose duty it would be to encourage abacus usage and education in all walks of life, much as has been role-modeled in Japan. It should be noted that multiplication and division can be performed also, as well as the extraction of roots, but their efficient operation is difficult to master without much time and practice. Such an organization could serve the purpose of developing such skills in a body of devotees, as well as developing a ranking and testing system, which could be correlated to skill levels used in Japan.

So, where does this leave the abacus? Have we seen its refinement complete, or are there more discoveries yet to be made using the bead frame calculator? Much discussion has occurred over the subject of refining abacus operations to *maximize speed*, by eliminating redundancy in both numbers of beads, and bead movement techniques. Conversely, implementation of the nine-bead approach tends to maximize *ease of operation*, by reducing the mental complexity inherent to the technique. Speed of operation is acquired, instead, by elimination of mental roadblocks that exist during the early stages of the learning process.

From a purely theoretical viewpoint, however, the refinement of the abacus is not complete. The author has investigated reducing the number of beads further by enacting changes to the way in which beads represent values. Consider the *binary nature* of the abacus bead. Being a digital device, each bead on the traditional abacus has only two states, active or inactive, slid to the bar or away from the bar. Changing the value of each bead may alter operation of the abacus, but the minimal number of beads required to represent values from zero through nine cannot be significantly reduced unless one first changes the binary nature of bead movements. The author has found that the abacus can be reduce to just *two* beads, if one allows for each bead to occupy *multiple positions*, with differing values in each position. In the two-bead abacus suggested, a single upper bead can occupy one of three positions, representing values of zero, three or six. The single lower bead can occupy one of four positions, representing values of zero, one, two or three. With these positional values, any number between zero and nine can be represented. Addition and subtraction can be performed similarly to the more conventional abacus, and complimentary methods can be used, based on a trinary system because of the bead values being in multiples of three.

Other variations on the two-bead abacus were also considered. For instance, when a soroban is being used, one normally looks at the group of beads pushed *toward* the bar as representing the numerical value in question. Consider, instead, the *space* where the beads are not. If one were to build an abacus that mimicked the movement of the space, two beads would be all that are required. One bead, above the bar, would represent zero and five. Another bead, below the bar, would occupy one of five positions: zero, one, two, three or four.

The physical nature of the abacus would have to change in order for any multi-positional version to be useful. Instead of a bead sliding on a rod or wire, one could employ a marble, sliding in a pocketed groove cut into the surface of a flat plate. Each pocket would provide a natural resting-place for the marble, yet it could be easily nudged into the next pocket higher or lower along that groove, thus providing the multiple-position capability required of such a device. In essence a flat plate or surface, such an abacus comes full circle to resembling the ancient calculating device of the Greeks and Romans. The author has constructed a prototype two-bead abacus, and finds its operation to be quite efficient, if one doesn't mind the change in mental process that is required in learning to operate such a device with efficiency. There is also a design, which incorporates a marble (or *calculi*) storage location drilled sideways into the thickness of the plate, with a sliding access cover to allow retrieval of calculi when necessary. Such a neo-roman abacus could truly be said to be portable. One could also imagine such abacus boards elegantly routered from solid slabs of fine hardwoods, with calculi of polished marble, or perhaps steel.

However, exploration of the minimal abacus does not end there. By increasing the number of bars available, the number of beads can also be reduced. Several four-bead variations have been found using two bars rather than the usual one. In both variants, there are two beads between the bars, each with a value of one. Two single beads reside above and below the pair of bars, with values of four and three, or five and two. Four beads can also be used if the number of bars is increased to three, with the bead values of one, two, three and four. By breaking the binary convention of bead value, the number of components can be reduced further. For instance, a four-bead abacus can use one bar, with a five-bead above and two one-beads and a single two-bead below, with their zero position in the middle, not touching a frame member. A three-bead abacus can be made using two bars. The upper bead represents four, the middle bead three, and the bottom bead has three possible positions: two, zero and one. There are several other variations possible on this theme.

How much further reduction can we perform on the abacus? Could one bead in each column fashion in a practical calculator? There would have to be ten positions available for the bead to reside in, zero through nine. To perform an operation such as three plus six, one would first slide the bead up three positions. Then, the bead would have to be slid up six more positions, yielding an answer of nine. The efficiency of such a device is no better than counting on one's fingers; the fingers have merely been replaced by positions along a column. Conversely, one could increase the number of bars to nine. Nine beads would be used, one under each bar, each with an increasing value from one to nine. Such a device could represent single quantities, but the operation on two or more values would be cumbersome, as the final value contained on any one column is obtained only be summing up the values of all the raised beads. Such a device does not come close to the definition of minimal.

In our discussion of the future of the abacus, mention also needs to be made of new arithmetic operations applied to the counting board. One example is base conversions. A fairly simple operation to perform is converting to and from binary. To represent a binary number, one need first to mentally recognize the base two place values of 1, 2, 4, 8, 16, etc., rather than the normal base ten place values. Secondly, a single bead is all that is needed to represent a one, while no bead entered represents a zero. Thus, the number nineteen is 10011 in binary. These conversion processes require the abacus frame be divided into a binary half and a decimal half, with the dividing point, as well as the place values, held in the operator’s memory. Simply making a decimal sum of all the binary place values that contain a one is all that is needed to perform the conversion from binary to decimal. In our example, the sum of sixteen, two and one is nineteen, the addition being performed on the “decimal half” of the abacus.

The conversion of a decimal number back to binary is also possible, the operation somewhat resembling division. Thus, in our example of nineteen, the largest power of two that fits is sixteen; a one is entered in the sixteen’s column. Sixteen is then subtracted from nineteen, leaving a three. The process is continued as we find two to be the largest power that fits into three. A one is entered in the two’s column, and two is subtracted from the three, leaving one. The process repeats until the answer 10011 is found.

Another arithmetic operation used by the Japanese on the soroban involves negative numbers. An example would be a merchant engaging in a transaction, where the total comes to $6.58. The customer offers a ten-dollar bill. What is the change required? Since the transaction was already performed on the abacus, the subtotal of $6.58 is already present on the frame. All the merchant needs to do is look at the beads that have *not* been moved on the active columns, and use them to represent a value, plus one. That would be the correct change. In this example, he finds “3.41” represents the beads *not* moved. Adding one to that value, the merchant finds that he owes the customer $3.42.

Negative numbers may also be encountered while summing up a column of financial figures, as in a checking account, for instance. How is one to proceed when this occurs? One simply “borrows” from an imaginary bead to the immediate left of the current subtotal and performs the subtraction as normal. The number represented is negative, whose value is again understood to be the beads *not* moved on the active columns, plus one. Should a subsequent quantity be added which makes the subtotal positive again, the “borrowed” bead is subtracted back out in order to balance the equation. One could assign the left-most column on the bead frame the duty of keeping track of the sign of the subtotal, by the presence of a raised bead, so that an interruption in the operation would not be fatal to the calculation.

VI. Conclusion

However minimal we may attempt to construct an abacus, or however complex an operation we may prove possible on the counting board, no operation is more simple, intuitive or natural than addition and subtraction on the nine-bead abacus, which the author has termed the *Nonus*, from the Latin meaning “ninth”. A person of minimal experience or dexterity can soon become adept at these two operations on the Nonus such that a pocket calculator becomes a hindrance. For by the time the “=” key is being pressed, the Nonus operator already has the answer before him, the calculation process being imbedded within the mechanism of data entry, as indicated earlier.

With the Nonus placed before him in use, the modern operator of the abacus becomes part of a connecting thread of human experience virtually as ancient as the race itself, spanning millennia and civilizations, cultures and the advance of history. The operator becomes part of a hidden triangle revealed, the vertices of which are the hand, the eye and the bead frame.