Octal Arithmetic on the Soroban

The soroban can be used for octal arithmetic in a similar manner to its use in decimal arithmetic. The octal or base-eight number system only uses the integers one through seven with the various rods representing powers of eight rather than powers of ten as in normal decimal arithmetic. There are two obvious methods for handling octal numbers on the soroban. The first is to continue the method we use for decimal arithmetic with the heaven bead counting five. We must never enter a number greater than seven on a rod, or if we do enter an eight or nine, we must then "normalize" or reduce it to a valid octal number it by adding two and carrying one to the left neighbor rod. The other method is to let the heaven bead count four and only use three earth beads. Here normalization is not necessary, but the numbers four, five, six, and seven are now represented by different bead configurations than we are used to. In my view, this is the preferable method and can be learned with only a little practice. The removal of the complications arising from normalization is worth the extra effort to learn the new bead configurations and complements.

Octal Arithmetic

Here are four methods to perform octal addition on the soroban: table look up, mental conversion of rods with numbers greater than seven, decimal addition followed by the addition of two to each rod which is greater than seven or from which a carry was generated, and octal addition with the heaven bead counting four and using only three earth beads. The first three of these use the normal decimal representation with the heaven bead counting five.

Table lookup

Octal addition table
(the leftmost 1 of a two-digit entry must be carried to the left neighbor rod)

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Octal subtraction table
(subtract the number in the top row from the number in the left column. An * indicates a borrow from the left neighbor rod)

-	0	1	2	3	4	5	6	7
0	0	*7	*6	*5	*4	*3	*2	*1
1	1	0	*7	*6	*5	*4	*3	*2
2	2	1	0	*7	*6	*5	*4	*3
3	3	2	1	0	*7	*6	*5	*4
4	4	3	2	1	0	*7	*6	*5
5	5	4	3	2	1	0	*7	*6
6	6	5	4	3	2	1	0	*7
7	7	6	5	4	3	2	1	0

Example: 12314 + 45664 = 60200

12314 Enter the first addend
From table, 1+4 = 5, replace 1 with 5
52314

From table, 2+5 = 7, replace 2 with 7
57314
From table, 3+6 = 11, replace 3 with 1 and add 1 to left neighbor
5(7+1)114
From table, 7+1 = 10, replace 7 with 0 and add 1 to left neighbor
(5+1)0114
From table, 5+1 = 6, replace 5 with 6
60114
From table, 1+6 = 7
60174
From table, 4+4 = 10, replace 4 with 0 and add 1 to left neighbor
601(7+1)0
From table, 7+1 = 10, replace 7 with 0 and add 1 to left neighbor
60(1+1)00
From table, 1+1 = 2, replace 1 with 2
60200

Subtraction can be done in a similar manner using the octal subtraction table.

Mental conversion of numbers greater than seven

As you add each number to a rod, if the sum is greater than seven, mentally subtract eight and carry one to the left neighbor rod (and check to see if that is now greater than seven).

Example: 12314 + 45664 = 60200

12314 Enter the first addend
1+4 = 5 not greater than seven, go to next rod
52314
2+5 = 7 not greater than seven, go to next rod
57314
3+6 = 9 greater than seven, subtract eight to leave 1 on this rod, carry makes left neighbor = 8 which is greater than seven, so subtract eight from that to leave 0 and carry to leftmost rod to make it 6
60114
1+6 = 7 not greater than seven
60174
4+4 = 8 greater than seven, subtract eight to leave 0 on this rod, carry to left neighbor makes it 8 which is greater than seven so subtract eight from that rod to leave 0 and carry to its left neighbor to make it 2
60200

Subtraction can be done in like manner, borrowing one from the left neighbor rod when needed and adding eight to the rod being operated on.

Decimal addition with two added for carries

Perform the addition on each rod as though it were normal decimal arithmetic, but if the addition on a rod yields a number greater than nine; that is, a carry is generated to the left neighbor rod using normal decimal arithmetic rules, then add an additional two to the rod generating the carry. An extra two must also be added to any rod which is greater than seven - this is best done as a separate step; that is, perform the entire addition, adding two only for the rods which generate carries, then go back and add two to any rods which are greater than seven. This last step may need to be performed more than once as carries may cause rods which were less than eight to become equal to or greater than eight.

Example: 12314 + 45664 = 60200

First pass

```
0012314
 + 4
-----------
0057314
 +  5
-----------
0062314
 +   2      add two because 7 + 5 generated a carry
-----------
0064314
 +    6
-----------
0057914
 +     6
-----------
0057974
 +      4
-----------
0057978
```

Second pass

```
0057978
 +    2     add two because rod is greater than 7
-----------
0058178
 +      2   add two because rod is greater than 7
-----------
0058180
```

repeat second pass because we still have some rods greater than seven

```
0058180
 +  2          add two because rod is greater than 7
-----------
0060180
 +     2       add two because rod is greater than 7
-----------
0060200        finished because all rods now are less than eight
```

Subtraction can be performed in a similar manner, but instead of adding two you must subtract two from a rod whenever a borrow is required from its left neighbor.

Octal addition with the heaven bead = 4

This is my preferred method, but takes a little practice to master. It's worth the effort because once learned it makes everything so much easier.

Let's label the bead settings as follows: H = heaven bead set (on the counter beam), E1 = a single earth bead set, E2 = two earth beads set, E3 = three earth beads set. We never have more than three earth beads set.. If we have the heaven bead and one earth bead set we can call this H, E1. If we have the heaven bead and two earth beads set, we can denote that by H, E2. If we have only three earth beads set, this is denoted by E3. No beads set is denoted by "none".

We will use the following bead configurations:

digit	beads set
0	none
1	E1
2	E2
3	E3
4	H
5	H, E1
6	H, E2
7	H, E3

8's complements
----
1 and 7
2 and 6
3 and 5
4 and 4

4's complements
----
1 and 3
2 and 2

Note: The comments in the following example simply reflect the mechanized method that I use for both decimal and octal arithmetic. The rules for this are to either push up or squeeze together the beads representing the number to be added, or if that's not possible, to pull down or spread apart the beads representing the complement of that number. If in this process the heaven bead is pushed up, carry to the left neighbor. Also carry to the left neighbor if the heaven bead is not moved but you pulled down the complement number on earth beads alone. There is no carry if the heaven bead was pulled down. Subtraction is the mirror image of this process.

Example: 12314 + 45664 = 60200

```
  1       2       3       1      4
 E1      E2      E3      E1     H
+ 4                                     pull down 4 (8's complement of 4) = H
----------------------------------------
  5       2       3       1      4
 H,E1    E2      E3      E1     H
        + 5                             squeeze together 5 = H and an earth bead
----------------------------------------
  5       7       3       1      4
 H,E1   H,E3     E3      E1     H
                + 6                     pull down 2 (8's complement of 6) = two earth beads
-----------------------------------------
  5       7       1       1      4      carry to left neighbor because heaven bead didn't move and we
 H,E1   H,E3     E1      E1     H           pulled down the complement of the number to be added
        +  1                            spread apart 7 (8's complement of 1) = H and three earth beads
------------------------------------------
  5       0       1       1      4      carry to left neighbor because heaven bead moved up
 H,E1   none     E1      E1     H
+ 1                                     push up 1 = one earth bead
-------------------------------------------
  6       0       1       1      4
 H,E2   none     E1      E1     H
                       +  6             squeeze together 6 = H and two earth beads
-------------------------------------------
  6       0       1       7      4
 H,E2   none     E1     H,E3    H
                               + 4      push up 4 = H
-------------------------------------------
  6       0       1       7      0      carry to left neighbor because heaven bead moved up
 H,E2   none     E1     H,E3   none
                       + 1              spread apart 7 (8's complement of 1) = H and three earth beads
--------------------------------------------
  6       0       1       0      0      carry to left neighbor because heaven bead moved up
 H,E2   none     E1     none   none
```

		+ 1			push up 1 = one earth bead

6	0	2	0	0
H,E2	none	E2	none	none

As you can see, this is straightforward abacus addition, but using different bead patterns than decimal arithmetic for 4, 5, 6, and 7 and using eight's complements rather than ten's complements. Once you have these bead settings and complements firmly in mind then with a bit of practice you should be able to calculate in octal using this method just as quickly as you can in decimal.

Conversion of a whole number from decimal to octal

One common method of converting integers from decimal to octal is to divide the decimal number by eight, taking the remainder as the first (least significant) digit of the octal number. The process is repeated with the whole number part of the quotient from the previous step being divided by eight and this remainder taken as the next most significant digit of the octal number. This process is continued until the quotient becomes less than eight, this last quotient being the most significant digit of the octal number. As an example of this, consider the conversion of the decimal number 567 to octal:

567/8 = 70, remainder 7 so 7 is the least significant digit of the octal number
70/8 = 8, remainder 6, so 6 is the next digit
8/8 = 1, remainder 0, so 0 is the next digit
1 is less than 8, so 1 is the most significant digit

567 decimal = 1067 octal

The problem with this method is that it requires repeated division. But there is a simpler way to do this conversion which does not require division; in fact, for decimal to octal conversion it requires nothing more than octal addition. This method is based on a general method which can be used to convert a whole number in any number base (or radix) to a number in any other base.

If we wish to convert an integer in a number system with base 'a' to a number system with base 'b' we can proceed as follows:
1. clear an accumulator
2. starting with the most significant digit of the base 'a' number to be converted, convert that digit to the number system with base 'b', then add it to the accumulator, performing the addition in the base 'b' number system.
3. multiply the entire accumulator by the base 'b' equivalent of 'a', performing the arithmetic in base 'b'.
4. take the next digit of the base 'a' number and repeat steps 2 and 3, continuing until the last digit of the base 'a' number has been added to the accumulator.

If we want to convert from decimal to octal, we are fortunate that we must multiply the accumulator by ten at each step, doing the multiplication in octal. In octal ten is written '12', so we can multiply the accumulator by nine, which is 11 in octal, and then add it to the original value of the accumulator. Multiplication by 11, of course, is trivial.

To do this conversion on the soroban, there is a multiplication technique called multifactorial multiplication which is ideally suited our needs. Using this method to multiply x times y we multiply y by x-1 and add that to y. Let's see how this works on the soroban by converting 567 decimal (base ten) to octal (base eight):

```
000 567 000 000  set the problem with the accumulator on the right end
           +       5  add the most significant digit of the decimal number to the accumulator
--------------------
000 067 000 005   and clear the decimal digit
           +      55  multiply the accumulator by ten (12 octal) using multifactorial multiplication
--------------------   (multiply by octal 11 and add)
000 067 000 062  the calculation is performed in octal !
           +       6  add the next digit of the decimal number and clear it from the decimal number
--------------------
000 007 000 070  addition in octal
             +  77    multiply the accumulator by ten (12 octal) using multifactorial multiplication
              +   00
--------------------
000 007 001 060  again, the calculation must be in octal
             +     7  add the last digit of the decimal number and clear it from the decimal number
--------------------
000 000 001 067  finished
```

Now let's try a slightly more complicated conversion.

Convert 6789 decimal to octal:

```
006 789 000 000  set the problem with cleared accumulator at the right
            +      6 add the first decimal digit
---------------------
000 789 000 006
             +    66  multiply by ten (12 octal) using multifactorial multiplication
---------------------
000 789 000  074
              +     7  add the second decimal digit
---------------------
000 089 000 103   multiply by ten (12 octal)
           +  1 1      (for simplicity, I don't show the result of each separate addition)
            +   00
              +   33
---------------------
000 089 001 236
             +    10  add the next decimal digit (8 decimal is 10 octal)
-------------------
000 009 001 246  multiply by ten (12 octal)
          + 11
           +  2 2
            +   44
             +    66
```

```
--------------------
000 009 015 174
            +    11  add the last decimal digit (9 decimal is 11 octal)
-------------------
000 000 015 205  finished
```

This method is easier than the successive division method, especially for larger numbers, if you are proficient at octal addition.

Conversion of a whole number from octal to decimal

We use the same general method to convert from octal to decimal, but in this case we are not so fortunate as to be able to multiply by simply replicating the digits and adding. We will still use multifactorial multiplication, but now we actually have to multiply each digit by seven in order to multiply the accumulator by eight. To make up for this extra effort, we get to do the addition in decimal. To summarize, we will take an octal digit, add it to the accumulator (no conversion necessary), then multiply the accumulator by eight using multifactorial multiplication (multiply by seven and add), then add the next octal digit, continuing in this manner until we are done. Let's work the previous example in reverse by converting the octal number 15205 to decimal.

```
015 205 000 000   set the problem
            +     1   add the first octal digit
--------------------
005 205 000 001
             +    7   multiply the accumulator by eight (decimal arithmetic)
--------------------
005 205 000 008
            +     5   add the next digit
-------------------
000 205 000 013   multiply the accumulator by eight
            +  07
             +   21
--------------------
000 205 000 104
            +     2   add the next digit
--------------------
000 005 000 106    multiply by eight
          + 0 7
            +  00
             +   42
--------------------
000 005 000 848
            +     0   add the next digit
--------------------
000 005 000 848
          +  5 6      multiply by eight
           +   28
             +   56
```

```
---------------------
000 005 006 784
            +      5   add the last digit
----------------------
000 000 006 789    finished
```

Conversion of a decimal fraction to an octal fraction

Enter the decimal fraction on the soroban and multiply by eight (decimal arithmetic). The overflow digit left of the decimal point is the next octal digit to append to the right side of the octal fraction. Clear the left of decimal overflow before multiplying by eight again to find the next octal digit. This multiplication can be done by using multfactorial multiplication - multiply by seven and add.

Example: Convert 0.795 to octal

```
000.795 000.000 000    set the problem
 + 4 9                 multiply by eight using multifactorial multiplication
  +  63
   +   35
-------------------------
006.360 000.000 000

000.360 000.600 000   clear the overflow and set it as the first octal digit
 + 2 1                multiply by eight
  +  42
-------------------------
002.880 000.600 000

000.880 000.620 000    clear the overflow and set it as the next octal digit
 + 5 6                 multiply by eight
  +  56
-------------------------
007.040 000.620 000

000.040 000.627 000    clear the overflow and set it as the next octal digit
 + 0 0                 multiply by eight
  +  28
--------------------------
000.320 000.627 000     overflow is zero

000.320 000.627 000     set the overflow (0) as the next octal digit
```

0.795 decimal = 0.6270 octal to four places

Conversion of an octal fraction to a decimal fraction

Enter the octal fraction on the soroban and multiply it by ten (octal arithmetic). In octal, ten is 12, so we can use multifactorial multiplication to multiply by 11 and add. The overflow left of the decimal point is the next digit to append to the right side of the decimal fraction. Clear the overflow before multiplying by eight again to find the next decimal digit. If the overflow is 10 or 11, it must be converted to decimal 8 or 9.

Example: Convert 0.627 to decimal

```
000.627 000.000 000   set the problem
 + 6 6                multiply by ten (octal arithmetic using multifactorial multiplication)
  +   22
   +    77
--------------------------
007.746 000.000 000

000.746 000.700 000   clear the overflow and set it as the next decimal digit
 + 7 7                multiply by ten
   +  44
    +   66
---------------------------
011.374 000.700 000

000.374 000.790 000   clear the overflow and set it as the next decimal digit (11 = 9)
 + 3 3                multiply by ten
  +   77
   +    44
---------------------------
004.730 000.790 000

000.730 000.794 000    clear the overflow and set it as the next decimal digit
 + 7 7                 multiply by 10
  +   33
---------------------------
011.160 000.794 000

000.160 000.794 900     clear the overflow and set it as the next decimal digit
```

0.627 octal is approximately 0.795 decimal

A real world use for octal arithmetic (in the USA) - carpentry in inches

Octal and hexadecimal are of course widely used in the computer field (calculation of memory addresses, for example) but here's a use for octal in carpentry. In the U.S., feet and inches are used to measure length and most rulers and tape measures have the inches divided into halves, quarters, eighths, and usually sixteenths of an inch. Measurements to the nearest thirty-second of an inch can be determined by reading between the lines. (For those of you on the metric system an inch is equal to 25.4 mm, so a sixteenth of an inch is about 1.6 mm and a thirty-second of an inch is about 0.8 mm.). Adding or subtracting a series of inch measurements, some in halves, quarters, or eighths and others in sixteenths or thirty-seconds can be difficult and prone to error. As an example, consider this calculation:

2 and 3/8 plus 1 and 9/32 plus 3/16 plus 1 and 7/8 plus 11/32 = ?

Of course, all of these could be converted to thirty-seconds, added together and the fractional part reduced by division or repeated subtraction to the simplest possible fraction. Here is a method which is more elegant, and in my opinion, easier.

* Select a unit rod for the whole inches - calculations on this rod and to the left which are all whole inches will be done in normal decimal arithmetic.
* The rod to the right of the unit rod will be used for eighths of an inch and calculations on this rod will take place using octal arithmetic with the heaven bead counting four and only three earth beads used.
* The rod to the right of the eighths rod will be used for sixteenths of an inch and calculations are in binary using only the heaven bead (heaven bead counts one).
* The rod to the right of this is used for thirty-seconds of an inch and calculations are in binary using the heaven bead just as for the sixteenths rod.

This is how it works on the example given above:

2 3/8 + 1 9/32 + 3/16 + 1 7/8 + 11/32 = 6 1/16 inches

(The unit rod for whole inches is the one shown just left of the decimal point below).

002.300 enter 2 3/8 (eighths rod has three earth beads set)

add 1 9/32
003.300 add 1 inch
003.301 9/32 is odd, so add 1/32, leaving 8/32 which is 4/16 or 2/8
003.501 add the 2/8 (eighths rod now has heaven bead and one earth bead set)
003.501 = 3 and 5/8 and 1/32 or 3 and 21/32 inches

add 3/16
003.511 3/16 is odd, so add 1/16, leaving 2/16 which is 1/8
003.611 add the 1/8 (eighths rod has heaven bead and two earth beads set)
003.611 = 3 and 6/8 and 1/16 and 1/32 or 3 and 27/32 inches

add 1 7/8
004.611 add 1 inch

005.511 add 7/8 in octal (eighths rod has heaven and one earth bead set)
005.511 = 5 and 5/8 and 1/16 and 1/32 or 5 and 23/32 inches

add 11/32
005.510 11/32 is odd, add 1/32, resetting the 1/32 rod and causing a carry
005.500 add the carry to the 1/16 rod, resetting it and causing a carry to the 1/8 rod
005.600 add the carry to the 1/8 rod - now we still have to add 10/32 = 5/16
005.610 5/16 is odd, so add 1/16, leaving 4/16 = 2/8
005.010 add the 2/8, clearing the 1/8 rod and causing a carry to the unit rod
006.010 add the carry to the unit rod
006.010 = 6 and 1/16 inches, the solution to our problem.

Conversion to/from Octal, Hexadecimal and Binary

There are methods to convert numbers directly between decimal and hexadecimal or binary which can be used on the soroban, but an easy way to do this is to use the methods for decimal/octal conversion described above in conjunction with the following table to convert between octal and hexadecimal or binary. Conversion between octal and hexadecimal is accomplished as follows: first use the table below to convert the octal or hexadecimal number to binary and set it on the soroban (one rod per bit using only the heaven bead), then, work outward from the decimal point, grouping the binary bits into groups of three to convert to octal or groups of four to convert to hexadecimal. If the number is too long to fit the binary representation onto the soroban, then the number can be converted twelve bits (four octal digits or three hexadecimal digits) at a time, beginning at the decimal point.

Octal	Binary	Hexadecimal	Binary
0	000	0	0000
1	001	1	0001
2	010	2	0010
3	011	3	0011
4	100	4	0100
5	101	5	0101
6	110	6	0110
7	111	7	0111
		8	1000
		9	1001
		A	1010
		B	1011
		C	1100
		D	1101
		E	1110
		F	1111

Example: convert the octal number 234567.2435 to hexadecimal

If we only have a thirteen rod soroban, we will have to take this four octal digits at a time, starting at the decimal point. Taking the octal fraction first:

2435 = 010 100 011 101 convert the octal digits to binary using the octal/binary table
= 0101 0001 1101 now consider the number as groups of four bits rather than three
= 5 1 D and convert to hexadecimal using the hexadecimal/binary table

Now do the same with the first four digits left of the decimal point

4567 = 100 101 110 111
= 1001 0111 0111 look at it as groups of four bits
= 9 7 7

and again with the next two digits, padding them with zeroes to make twelve bits

0023 = 000 000 010 011
= 0000 0001 0011
= 0 1 3

so 234567.2435 octal = 13977.51D in hexadecimal
and = 10011100101110111.010100011101 in binary

In like manner we can convert from hexadecimal to octal by setting the binary on the soroban, taking three hexadecimal digits at a time starting from the decimal point and regrouping these twelve binary bits into four groups of three. The corresponding octal numbers can then be looked up in the table.

Steve Treadwell
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