Easy Division Method for Soroban/Abacus

Here is an easy method for division on the soroban or abacus (perhaps better suited to sorobans as they typically have more rods than the Chinese suan pans). The method was developed in an attempt to make division easy by reducing mental multiplication as much as possible in favor of comparison and subtraction and also by the simplification of quotient estimation. The method does require an extra step of computing the divisor multiplied by five, but this is a trivial calculation, as described in Note 2 below.

The method is fundamentally long division with repeated subtraction of the divisor from the current working dividend fragment, but with improvements over the totally brute force method. Having available the 5x multiple of the divisor means we can do one subtraction instead of five if the quotient digit is five or greater. Equally important, it ensures that for quotient digit estimation (deciding how many times the divisor will divide into the current dividend fragment) we never need to consider any multiple higher than four times the divisor. This also makes it easier to make the estimation by using an “estimation divisor” which is greater than the actual divisor and for which it is very easy to mentally compute the 2x, 3x, and 4x multliples, while at the same time ensuring that we never undershoot the quotient digit by more than 1. An additional benefit is that having the 5x multiple of the divisor, we also have available at a glance one half of the divisor which makes it a trivial matter to round off the quotient to a certain number of significant figures.

The procedure after setting the divisor and dividend and calculating the 5x multiple of the divisor is, as in normal long division, to begin with the first dividend fragment which is greater than or equal to the divisor. Now compare this fragment to the 5x multiple of the divisor; if it is larger than the 5x multiple, set the appropriate quotient digit to five and subtract the 5x divisor multiple from the dividend fragment to obtain a new dividend fragment. Now that we know that the divisor will divide into our current dividend fragment a maximum of four times, instead of trying to do mental division with the actual divisor to estimate the quotient digit (to be added to the zero or five already set) it is easier to use a number greater than the divisor but which is much easier to mentally multiply by two or three or four. Using a number greater than the divisor ensures that the quotient digit we estimate will never be too large and we will never subtract too much from the dividend fragment and thus get a negative number. By careful choice of the number we use for the estimation divisor we can also ensure that we never undershoot the quotient digit by more than one. The divisor (normalized to between one and ten) will fall into one of the ranges in the following table. For a divisor in a particular range, use the associated estimation divisor. For example, if the divisor is 372.6, test whether 400.0 will divide into the dividend fragment once, twice, three times, four times, or not at all. Use this number as the quotient digit and subract the divisor from the dividend fragment that many times. Then check to see if the dividend fragment is now less than the divisor. If not, increment the quotient digit and subtract the divisor once more. When the dividend fragment has been reduced to less than the divisor, append a digit from the dividend to the the right end of the dividend fragment to make a new dividend fragment and compute the next quotient digit as above, beginning by comparing the new fragment to the 5x multiple of the divisor.

For divisors between	Use this estimation divisor
1 to 1.25	1.25
1.25 to 1.5	1.5
1.5 to 2	2
2 to 2.5	2.5

2.5 to 3	3
3 to 4	4
4 to 5	5
5 to 6	6
6 to 7	7
7 to 8	8
8 to 9	9
9 to 10	10

Derivation of these numbers is described in Note 3 below.

A couple of examples will help in understanding the method.

Example 1: 8710.8 / 11.9 = 732

Set the problem – divisor on left end of the soroban, dividend in the middle, 5x the divisor on the right end.

```
          U                 place dividend left of a unit rod by 4-2-2 = 0 rods
                                (dividend digits – divisor digits - 2)
119xxxxx0087108xxxxxx595
         -595               subtract 5x divisor and set quotient digit = 5
119xxxxx5027608xxxxxx595
                            estimation divisor is 125 and divides into dividend
                                fragment 276 twice, so add 2 to quotient and
                                subtract divisor twice
       +2
         -119
         -119
119xxxxx7003808xxxxxx595    finished with this quotient digit
                            new dividend fragment 380 is less than 5x divisor, no
                                5x subtraction
                            estimation divisor 125 divides into dividend fragment 3
                                times, so set quotient digit = 3 and subtract
                                divisor 3 times
        +3
          -119
          -119
          -119
119xxxxx7300238xxxxxx595    finished with this quotient digit
                            new dividend fragment 238 is less than 5x divisor, no
                                5x subtraction
                            estimation divisor 125 divides into dividend fragment
                                once, so set quotient digit = 1 and subtract
                                divisor
         +1
           -119
119xxxxx7310119xxxxxx595    undershot quotient digit by 1, increment & subtract
         +1
           -119
119xxxxx7320000xxxxxx595    finished – least significant digit of quotient is on
          U                     the unit rod
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Example 2: 2908108 / 734 = 3962

Set the problem – divisor on left end of the soroban, dividend in the middle, 5x the divisor on the right end.

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          U                 place dividend left of a unit rod by 7-3-2 = 2 rods
734xxx002908108xxxxx3670    734 doesn’t go into 290, so add a digit to the dividend
                               fragment and start the quotient next to the dividend
                            dividend fragment 2908 is less than 5x divisor,  no
                                5x subraction
                            estimation divisor 8 divides into 29 3 times, so set
                                quotient digit = 3 and subtract divisor 3 times
      +3
        -734
        -734
        -734
734xxx030706108xxxxx3670    finished with this quotient digit
                            new dividend fragment 7061 is greater than 5x divisor
       +5                   set quotient digit = 5
        -3670               subtract 5x divisor
734xxx035339108xxxxx3670
                            estimation divisor 8 divides into 33 4 times
       +4                   add 4 to the quotient digit
         -734               subtract the divisor 4 times
         -734
         -734
         -734
734xxx039045508xxxxx3670    finished with this quotient digit
                            new dividend fragment 4550 is greater than 5x divisor
        +5                  set quotient digit = 5
         -3670              subtract 5x divisor
734xxx039508808xxxxx3670    estimation divisor 8 divides into 08 1 time
        +1                  add 1 to the quotient digit
          -734              subtract the divisor once
734xxx039601468xxxxx3670    finished with this quotient digit as 146 < divisor
                            new dividend fragment 1468 is less than 5x divisor
                            estimation divisor 8 divides into 14 one time
         +1                 set quotient digit = 1
           -734             subtract divisor once
734xxx039610734xxxxx3670    dividend fragment > or = divisor indicates undershoot
         +1                 add one more to the quotient digit
           -734             and subtract the divisor once more
734xxx039620000xxxxx3670    finished – least significant digit of quotient is on
          U                     the unit rod
```

Rounding off:
In the above example, if we only wanted the quotient to three significant digits, we could have rounded off the solution after computing just three quotient digits. Back up to the point where we have just determined the third quotient digit:

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734xxx039601468xxxxx3670
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To determine whether or not the quotient should be 3960 or rounded up to 3970, we just need to compare the next dividend fragment, 1468, to the 5x divisor multiple, 3670 (actually we are comparing the remainder 146.8 to half the divisor 367.0). Because 1468 is less than 3670, we don’t round up the third digit of the quotient, so the solution, to three significant digits, is 3960.

Note 1. When doing multiple subtractions of the divisor, I find it easier to perform all the subtractions on one rod, then move to the right neighbor rod and do all the subractions, then the next rod, etc. So, for example, if I had to subtract 734 three times, I would subtract 7 three times, then move right one rod and subtract 3 three times, then move right again and subtract 4 three times. This seems both easier and faster to me, but of course you can subtract 734 three times if you prefer. One way to shorten the process of multiple subtractions which can frequently be used, is to mentally double the divisor and subtract the double; this way, two subtractions can be reduced to one, three subtractions to two, and four subtractions to two. For long divisors which are not easily mentally doubled, it's probably best just to do all the subtractions.

Note 2. Calculation of the divisor multiplied by five: place the divisor at the far right end of the soroban, except for the very right-most rod which remains cleared to zero. Using this far right-most rod as the least significant digit, we have multiplied the divisor by ten and now must divide by two. Starting at the right-most rod, proceed as follows:

1. If the number on the rod is even, go to step 2, else decrement the number to make it even and add five to the right neighbor rod by moving its heaven bead to the counter beam. Now go to step 2.
2. Subtract half of the count on the rod
3. Move one rod to the left and repeat steps 1 and 2 – continue until all non-zero rods have been processed – we now have 5x the divisor at the right end of the soroban.

Note 3. Determination of "estimation divisors"

The object is, for any divisor, to find another number larger than the actual divisor to use as a trial divisor to find an estimated quotient digit. The estimation divisor should be a number whose 2x, 3x, and 4x multiples can be instantly computed without effort. Using an estimation divisor as large or larger than the actual divisor ensures that we will never overshoot the quotient digit, so when we multiply the actual divisor by the quotient digit derived from the estimation we will never have a product larger than the current dividend fragment from which this product is to be subtracted. We will never have to revise the quotient downward, but of course, we may need to revise upward. But we don't want to need revision by more than one, as each revision requires another subtraction of the divisor from the dividend fragment. The problem is, for any given divisor, to determine the best estimation divisor.

We know that, having already subtracted, if necessary, five times the divisor from the dividend fragment, the actual divisor and estimation divisor will only divide into the dividend fragment a maximum of four times. How much larger than the actual divisor can we make the estimation divisor without danger of estimating a quotient digit smaller than the actual quotient by more than one? Let d = the actual divisor, e = the estimation divisor, q = the actual quotient digit, and D = the dividend fragment. The worst case situation is when the divisor divides into the dividend fragment exactly q times: D = q*d. How much larger can e be relative to d so that e divides into the dividend fragment exactly q-1 times (so that if e were any larger, we would estimate q-2); that is to say, where D = (q-1)*e ?
We have D = q*d and D = (q-1)*e, so q*d = (q-1)*e or e = d*(q/q-1). We know that q is a maximum of four, so we could have e = (4/3)*d, or (3/2)*d, or (2/1)*d. The smallest of these is 4/3 and this will ensure that we won't undershoot the quotient by more than one. Note that if we hadn't first subtracted the 5x multiple of the divisor (when necessary) the quotient digit could be any number up to nine and we would be limited to an estimation digit of no more than 9/8 of d.

So now that we know that our estimation divisor can be a maximum of 4/3 the actual divisor without undershooting the quotient digit by more than one, how do we choose a good set of estimation divisors? We could start at 1 and just multiply by 4/3, so the estimation divisor for all divisors between 1 and 1.333... would be 1.333..., for all divisors between 1.333... and 1.777… would be 1.777…, etc. These are not numbers for which we can easily compute the 2x, 3x, and 4x multiples, but we can pick any smaller number, so let’s use these estimation divisors which are easy to multiply by 2 or 3 or 4 and are still less than or equal to 4/3 times the minimum actual divisor in the corresponding range (all normalized to between one and ten – shift the decimal point as required by the problem):

For divisors between	Use as estimation divisor	Est. divisor/min. divisor
1 and 1.25	1.25	1.25/1 = 1.25
1.25 and 1.5	1.5	1.5/1.25 = 1.20
1.5 and 2	2	2/1.5 = 1.333…
2 and 2.5	2.5	2.5/2 = 1.25
2.5 and 3	3	3/2.5 = 1.20
3 and 4	4	4/3 = 1.333…
4 and 5	5	5/4 = 1.25
5 and 6	6	6/5 = 1.20
6 and 7	7	7/6 = 1.166…
7 and 8	8	8/7 = 1.143…
8 and 9	9	9/8 = 1.125
9 and 10	10	10/9 = 1.111…

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