立方根 Cube Roots (contributed by Shane Baggs)

This web page was put together by members of the SorobanAbacus group when we realized that the contemporary Japanese cube root method for soroban didn't seem to be published anywhere. We started with the 19th Century Japanese method[1], adjusted it to conform to modern soroban techniques[2], then took suggestions from group members who know the method to make it look more like the way it's done today. The result is something very close to the contemporary Japanese method, but which also resembles the method used in The Nine Chapters and the Horner Method.

To take a cube root, you'll need to know the cubes of the digits.

Number		Its Cube
1	---------	1
2	---------	8
3	---------	27
4	---------	64
5	---------	125
6	---------	216
7	---------	343
8	---------	512
9	---------	729

Example 1: The cube root of 51,064,811 is 371.

The cube root method is unique in that there is a "stopping place" on the abacus where you stop dividing and leave a remainder. The remainder gets cleaned up in a later step where you reverse the division by multiplying a number back into it. You have to keep track of which numbers are which in order to avoid going too far. The example will make this clear.

```
Set up the problem.
[ .  .  .  .  .  .  . ]
ABCDEFGHIJKLMNOPQRSTUVW
00000000000000051064811  Divide the number into groups of three rods.
--.--.--.--.--.--.--.--  Leave an extra group of three rods to the left
|          |             for the answer.  The answer will start on the
|          answer        leftmost rod of this extra group.  The extreme
triple of answer         left of the soroban will hold the triple of
.                        the answer.
```

```
Get the first digit of the answer.
Subtract the cube.
ABCDEFGHIJKLMNOPQRSTUVW
00000000000300051064811  Look at the first group.
.             -27        What number cubed goes into 51?
00000000000300024064811  3 cubed is 27.
```

The answer tripled is 3x3=9. Place this on rods AB on the far left.

```
Update the triple.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000300024064811
```

That was the first cycle.

To start the second cycle:

Second cycle

```
Divide by the triple.
ABCDEFGHIJKLMNOPQRSTUVW  Now bring the second group into it.
09000000000300024064811  Divide everything up to the end of this
.              /9        second group by the triple, 9.
09000000000300267307811  Don't go too far!
--         -  ---- -     Stop as soon as you deal with rod T.
|          |  |    |
|          |  |    The 9's remainder 7 remains on rod T.
|          |  The 9's quotient is 2673.
|          The answer so far is 3.
Its triple is 9, which is what we divided by.
```

To get the next digit of the answer, look at the quotient without its last digit, which is 267. 3_ x _ = 267? 8x3=24, so 8 fits.

```
Get the next digit.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000380267307811
```

We now multiply 38 x 8 and subtract it from 267. Because the leftmost rod A has a zero on it, we have to shift everything one place to the right, which you can do by imagining that we're multiplying 038 x 8. I will give an explanation
at the end of why things are different when the leftmost rod on the soroban is empty.

```
Multiply and subtract.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000380267307811
.           -00          8 x 0 = 0
.            -24         8 x 3 = 24
09000000000380027307811
.             -64        Oops!  You can't take 64 from 27, so the digit I picked
.                        must be too big.  I'll revise the 8 to a 7 and continue.
.          -1+03         As in division revision, subtract 1 from the answer
09000000000370057307811  and add 03 to the number I'm dividing.
.             -49
09000000000370008307811
--         -- ---- -      We wound up multiplying 37 x 07.
|          |  |    |
|          |  |    9's remainder
|          |  9's quotient
|          answer
triple
```

We now have a board with a remainder on it.

For the next step, we must use multiplication to reverse the step where we divided by the triple, and fold the quotient back into its remainder.

Why is the triple still 9 when the answer is no longer 3 but 37?

We cannot update the triple yet because this step must EXACTLY REVERSE the one where we divided by 9, therefore the triple must remain 9. As we won't be ready for the updated triple until the next cycle, we will update the triple as the last step of this cycle.

We fold the 9's quotient back into the 9's remainder by multiplying 83 by 9, and adding it to whatever 9's remainder is already on column T (it's 7 this time.)

```
Reverse.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000370008307811
.               x9       83 x 9 + 7 = 754
09000000000370000754811
```

There are only two more steps left in the second cycle, and they're a lot like the last two steps of the first cycle.

We take the second digit of the answer that we found in this cycle, and subtract its cube from the second group of digits on rods RST.

```
Subtract the cube.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000370000754811
.               -343     7 cubed is 343.
09000000000370000411811
```

To finish the second cycle, we update the triple by adding the triple of 7 to rods BC.

```
Update the triple.
ABCDEFGHIJKLMNOPQRSTUVW
09000000000370000411811
+21                      3 x 7 = 21
11100000000370000411811
```

Now that the triple has been updated, we may start the third cycle.

The third cycle begins by bringing in the third group on rods UVW. It proceeds as the second cycle.

We start by dividing by the triple of the answer so far, 111, stopping when we deal with the last rod.

Third cycle

```
Divide by the triple.
ABCDEFGHIJKLMNOPQRSTUVW
11100000000370000411811
.               /111
11100000000370037100001
---         --  ---- ---
|           |   |    |
|           |   |    111's remainder
|           |   111's quotient
|           answer
triple
```

Now, take the 111's quotient 3710 without its last digit, or 371.

37_ x _ = 371? 1 fits perfectly.

```
Get the next digit.
Multiply and subtract.
ABCDEFGHIJKLMNOPQRSTUVW
11100000000371037100001
.             -371
11100000000371000000001
```

Notice that when we multiplied 371 x 1, we didn't have to shift things to the right by pretending we were multiplying 0371 x 1. That's because the leftmost rod of the soroban is no longer empty, so we need not observe the exception. Again, there will be an explanation at the end of why this exception is necessary.

Here we would normally reverse the division step by multiplying the 111's quotient by 111 again and adding it to the 111's remainder, but since there's nothing left of the 111's quotient but zero, and 0 x 111 = 0, we can skip this step.

(Reverse.)

We now perform the next-to-last step of subtracting the cube of the third digit we just found, 1, from the third group on UVW.

```
Subtract the cube.
ABCDEFGHIJKLMNOPQRSTUVW
11100000000371000000001
.                    -1  1 cubed is 1.
11100000000371000000000
```

Normally we'd perform the final step of updating the triple here, but updating the triple just gets things ready for the next cycle. Since we're done and the problem has come out even, we can stop here.

(Update the triple.)

The answer is 371, and it comes out even with no remainder.

WHY ARE THINGS DIFFERENT WHEN THE LEFTMOST ROD OF THE SOROBAN IS EMPTY?

One of the unique things about the cube root is that you divide by the triple of the answer, perform some operations, then reverse that step by multiplying by the triple. The operations must exactly mirror one another.

In 19th Century Japan, rather than using the triple of the answer, they would first divide by the answer and then by 3. To reverse this, they would multiply by 3 and then by the answer. This was slower, because for each step we take today they would have to have taken two steps. However, there was an advantage: no exceptions!

The numbers they were dividing by were either static (the 3) or would always grow by 1 digit per cycle (the answer), so the number of places the unit rod would walk to the left would always increase by exactly one rod per cycle. Combined with all the other movement going on (the stopping point, the answer itself) the result was that you could place the answer once and it would fit for the duration. There were no exceptions.

Today we speed things up by multiplying the answer and the 3 together into a triple; then dividing and multiplying by this triple. In most cases this is just as predictable as the answer and the number 3 taken separately. Usually, the first group is at least 64, its cube root is at least 4, and its triple is at least 12; this yields a 2-digit triple that grows by one digit each cycle. We position the answer for this case as this is the most common, and so long as these conditions hold, everything works.

However, if the first group is any smaller than 64, its cube root is 1, 2, or 3; and its triple is 3, 6, or 9 -- all single-digit numbers when we're expecting 2-digit numbers. When we are dividing by a number with one less digit than what we're ready for, the unit rod walks one too few places to the left and the quotient winds up one place too far to the right.

Can we compensate for this by placing the answer one place further to the right when the triple is a single digit? Not always. In the example above, the answer was 371. In the second cycle the triple was 9. Had we placed the answer one place to the right from where it usually goes, that would have made the second cycle work. However, in the third cycle the triple is 111, which has the number of digits we would expect. Because the triple "fixed" itself by growing by two digits in one step, the answer that is in the right position for the second cycle is in the wrong position for the third. There is no position that will answer for both cycles. That is why we position the answer for the most common case.

The logical way to deal with this is to place the triple on rods AB. That turns rod A into a marker -- an empty rod A means we're dealing with an exceptional case. Rod A was empty in the second cycle and not empty in the third.

The obvious way to deal with the exception is to move everything one extra place to the left when dividing by the triple, then one extra place to the right when multiplying by the triple. The alternative is to move everything one place to the right during the step that comes between dividing by the triple and multiplying by the triple.

The alternative is better because it only involves changing one step per cycle rather than two. This step is shorter, and it is always multiplication, in which imagining multiplying a leading zero achieves the desired effect. It is therefore this solution which I present above.

Example 2: The cube root of 67,917,312 is 408.

While this example can be worked using the same method as the first example, I would like to demonstrate a shortcut that you can do when you encounter a zero in the answer.

First cycle

Set up the problem, dividing it into groups of three rods.

```
Set up the problem.
[ .  .  .  .  .  .  . ]
ABCDEFGHIJKLMNOPQRSTUVW
00000000000000067917312

Get the first digit.
Subtract the cube.
Update the triple.
00000000000400067917312  Look at the first group of rods, OPQ.
.               -64      4 cubed is 64 which goes into 67.
00000000000400003917312  Triple the answer so far (4 x 3 = 12)
.                        and put it on rods AB.
12000000000400003917312
```

Second cycle

```
Divide by the triple.
12000000000400003917312  Bring in the next group of rods, RST,
.                  /12   and divide by 12.
12000000000400326005312  326 is the quotient, 5 is the remainder.

Get the next digit.
Consider the quotient 326 without its last digit, or 32.
4_ x _ = 32?  The answer is zero.
```

WHEN YOU ENCOUNTER A ZERO IN THE ANSWER you can take this shortcut:

Finish the current cycle early by mentally placing a zero after the triple, just as you mentally placed a zero after the answer so far.

Start the next cycle by bringing in the next group of three rods.

Divide by the triple, but treat all division up to and including the first rod of the new group of three rods as if it had already been done.

Why does it work? See the section below on WHY CAN YOU USE A SHORTCUT... where I work it out the long way.

Third cycle thanks to the shortcut.

```
Continue to divide by the triple.
ABCDEFGHIJKLMNOPQRSTUVW  The answer so far is 40, and its triple is 120.
12000000000400326005312  I have brought in rods UVW.
.                   /120  I divide by 120 as if I have already been
.                +4-480   dividing by 120 up to and including column U.
12000000000400326400512  Now it's done up to and including column V.
.                 +4-480
12000000000400326440032  Now it's done up to and including column W.

Get the next digit.
Consider the 120's quotient 32644 without its last digit, or 3264.
40_ x _ = 3264?  32/4=8, so the answer is 8.

Multiply and subtract.
Reverse.
Subtract the cube.
12000000000408326440032  Subtract 408 x 8 from 3264.
.             -32
.               -64
12000000000408000040032  Reverse the division by 120.
.               x120      4 x 120 + 32 = 512.
.                 -4+480
12000000000408000000512
.                   -512  Subtract the cube of 8, or 512, from rods UVW.
12000000000408000000000  The answer is 408, and it comes out even.
```

WHY CAN YOU USE A SHORTCUT WHEN THE ANSWER CONTAINS A ZERO?

I'll demonstrate that this works by backing up the step before I mentioned the shortcut and working it the long way to show that the abacus does indeed return to this state.

Back to the Second cycle

```
Get the next digit.
Multiply and subtract.
ABCDEFGHIJKLMNOPQRSTUVW  BACK TO THE SECOND CYCLE
12000000000400326005312  The answer is zero.  Subtract 40x0=0 from 32.
                         (Nothing happens.)
Reverse.
12000000000400326005312  Reverse the multiplication step by multiplying
.             x12        326 by 12 and adding it to the remainder 5.
12000000000400003917312

In this case it really did REVERSE the multiplication step completely,
and we're back to an even earlier step!

Now, subtract the cube of 0 (which is 0) from rods RST.
This does nothing.

Subtract the cube.
12000000000400003917312

Update the triple by adding a zero to the end of 12, making 120.

Update the triple.
12000000000400003917312
```

We now start off the THIRD CYCLE with the abacus in the same state it was at the beginning of the second cycle. It's the same except for the mental zeros that tell us that the answer is 40 and the triple is 120, rather than the answer being 4 and the triple being 12.

Third cycle

Bring in rods UVW and divide by 120.

```
Divide by the triple.
ABCDEFGHIJKLMNOPQRSTUVW
12000000000400003917312
                /120
.             +3-360
12000000000400300317312  Division by 120 is complete to (and including) rod S.
.              +2-240
12000000000400320077312  Division by 120 is complete to (and including) rod T.
.               +6-720
12000000000400326005312  Division by 120 is complete to (and including) rod U.
```

Do you recognize the state of the abacus? It's now in the same state it was in cycle 2 when we realized that the second digit of the answer was zero. In the second cycle we divided by 12 up to rod T, which as you can see looks the same as dividing by 120 up to rod U. The shortcut takes us this far.

If we continue to divide by 120,

```
ABCDEFGHIJKLMNOPQRSTUVW
12000000000400326005312
.                +4-480
12000000000400326400512  Division by 120 is complete to (and including) rod V.
.                 +4-480
12000000000400326440032  Division by 120 is complete to (and including) rod W.
```

and work proceeds as above.

References

[1] Knott, Dr. Cargill G.,
"The Abacus, in its Historic and Scientific Aspects,"
Transactions of the Asiatic Society of Japan (Yokohama, R. Meiklejohn & Co., 1886): 18-71.

[2] Kojima, Takashi,
The Japanese Abacus, its Use and Theory
Tokyo, Charles E. Tuttle Co., 1954.

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