THE ABACUS, IN ITS HISTORIC AND SCIENTIFIC ASPECTS.

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PART I.—THE HISTORIC ASPECT.

The *Soroban* (算盤) or Japanese Abacus is one of the first objects that strongly attracts the attention of the foreigner in Japan. He buys at some shop a few trifling articles and sums up the total cost in his own mind. But the tradesman deigns not to perplex himself by a process of mental arithmetic, however simple. He seizes his *Soroban*, prepares it by a tilt and a rattling sweep of his hand, makes a few rapid, clicking adjustments, and names the price. There seems to be a tradition amongst foreigners that the *Soroban* is called into requisition more especially at times when the tradesman is meditating imposition; and in many cases it is certain that the Western mind, with its power of mental addition, regards the manipulator with a slight contempt. A little experience, however, should tend to transform this contempt into admiration. For it may be safely asserted that even in the simplest of all arithmetical operations the *Soroban* possesses distinct advantages over the mental or figuring process. In a competition in simple addition between a "Lightning Calculator," an accurate and rapid accountant, and an ordinary Japanese small tradesman, the Japanese with his *Soroban* would easily carry off the palm.

It is true that the Japanese often uses his board and beads when the operation is simple enough to be completed mentally during the time that he stretches his hand out to take hold of his instrument; but that is only an illustration of the irresistible force of habit. To him the mention of any arithmetical operation suggests "Soroban." He could no doubt, if he tried, add 12 and 18 in his mind; but before he has time to recognise the peculiar simplicity of any special problem, and, dispossessing his thought of "Soroban," proceed to solve it as the foreigner does, he would waste more time in mental labour than is expended in the manual labour of adjusting and manipulating his counters. The only blame indeed that can be attached to him for using his instrument to add 5 to 8 is that he is strictly consistent. But let us suppose that a purchaser has bought three articles which are priced at Yen 1.25, Yen 2.89, and Yen 3.17 respectively. How many people out of any hundred of ordinary intellect could add these three numbers correctly in their mind? A Japanese shop-boy with *Soroban* in hand will do it as fast as the numbers can be named, and with greater precision and certainty than many of us could attain in figuring. Facts like these suffice to give to the instrument a certain respectability.

The Abacus possesses besides a high respectability, arising from its great age, its wide-spread distribution, and its peculiar influence in the evolution of our modern system of arithmetic. In the Western lands of to-day it is used only in infant schools, and is intended to initiate the infant mind into the first mysteries of numbers. The child, if he ever is taught by its means, soon passes from this bead-counting to the slate and slate pencil. He learns our Indian Numerals, of which *one* only is at all suggestive of its meaning; and with these symbols he ever after makes all his calculations. In India and all over civilized Asia, however, the Abacus still holds its own; and in China and Japan the method of using it is peculiarly scientific. It seems pretty certain that its original home was India, whence it spread westward to Europe and eastward to China, assuming various forms, no doubt, but still remaining essentially the same instrument. Its decay in Europe can be traced to the gradual introduction and perfecting of the modern cipher system of notation, which again in part owes its early origin to the indications of the Abacus itself. According to the results arrived at by Sir E. Clive Bayley, in his discussion of the genealogy of modern numerals,[1] the main facts seem to be these. The Abacus finds its earliest historic home in India, where originally it existed alongside of most complicated systems of numerical notation. The gradual simplification of these in accordance with the universal tendency of the human mind under civilisation—a simplification which largely consists in borrowing from elsewhere—brought them into closer and closer correspondence with the indications of the Abacus. At last with the evolution of the *zero*, the notation became accurately symbolic of the columns of the Abacus, and rapid calculation was possible without their aid. In Europe the new system, introduced through the Arabs, gradually displaced whatever "counter" system was in vogue. But the substitution of the symbolic for the mechanical was only partial in India, while in China and Japan centuries have been insufficient to effect the change. These facts are sufficient to show that the ciphering system is not so very superior to the Abacus as we of western training are apt at first to imagine. That the Chinese and Japanese should still use an instrument, which to us is suggestive of an infant school, is startling. To explain it as a result of the general conservatism of the eastern mind is nothing to the point; for not only has the conservatism itself to be explained, but we have in the non-conservative character of the Japanese mind a fact that cannot be disregarded. I think the true explanation is to be found in the processes of natural selection, which of course vary with the mental habit of the race. The problem is twofold. What causes, not present in the East, led to the ascendency of ciphering over bead-counting in the West; and do these causes imply any difference in the mental attitudes of the peoples? It is convenient to discuss these questions under two heads.

First I shall consider comparatively the systems of numerical notation that have been invented amongst civilised peoples, and then proceed to compare the systems of numeration or nomenclature of numbers. I have placed notation first, not because of any logical necessity, but because of its greater simplicity. Speaking of course

[1] Journal R. A. S., vols. xiv, xv, xvi.

precedes writing; but that hardly implies that numerical notation necessarily succeeds number-naming. It is quite conceivable that man should have indicated a number graphically or pictorially before he had a name for it. We often hear of the savage who cannot *number* beyond two or three or five, which usually means that he has no names for numbers above that limit. But to infer that he cannot reckon beyond that limit is certainly illogical. The remark made by a native to Dr. Koelle,[2] at the time missionary in Sierra Leone, is quite to the point here. Dr. Koelle expressed surprise that they should be able to do in daily life with numerals only to the limit of five, to which one replied:—"We can manage very well; for having counted five, we put it aside on one heap and then begin another, and so on, as many as we want." The same is found amongst the natives of the New Hebrides, who count off by bundles of ten, and use the same word for forty as for four, making up by gesture for lack of language. And a moment's thought will show that we ourselves do exactly the same, only that we give *names* to our bundles of ten, a result probably of the development of writing. Notation indeed has quite outstripped nomenclature; and nomenclature itself may ultimately depend upon notation, used in its widest sense of pictorial symbolising. To this point we shall return later.

The graphic representation of numbers may be traced historically through four well-marked stages, which I shall call for ease of reference the Pictorial, the Symbolic (including the Alphabetic), the Decimal, and the Cipher stages. These names are not to be taken in too literal a sense; and we must remember that in many classifications it is difficult exactly to draw the lines of demarcation between the classes—each one partaking more or less of the special characteristics of the others. Thus we have Pictorial numerals up to four in the Roman system, and to three in the Chinese; but the Roman belongs distinctly to the Symbolic stage, and the Chinese to the Decimal. In a loose sense the term Decimal applies to both Symbolic and Cipher systems; but here it is, for the sake of greater definiteness, restricted to those systems which have a distinct symbol for *ten* and repeat it in the higher numbers.

[2] Journal R. A. S., vol. xvi.

It may be stated at once that there are no examples amongst civilised nations of a purely Pictorial System. In the early Egyptian Hieroglyphics, *one* was represented by a vertical or (more rarely) a horizontal, line; and by repetitions of this, the other numbers were figured up to *nine*. Such a system, however, could hardly be carried much further without giving rise to confusion. Even the *eight* () and *nine* () would be apt to be mistaken for each other; and higher combinations of course still more so. Also, as writing became more widely used, a necessity arose for shortened processes. Hence, as a result of the desire to save time and prevent misunderstanding, a peculiar symbol for *ten* was evolved, in shape somewhat resembling a croquet hoop. This symbol was then used in obvious pictorial combinations to represent 20, 30, 40, etc., up to 90. With the aid of these two symbols numbers up to 99 were figured. Higher numbers were represented with the aid of other peculiar symbols for 100, 1000, 10,000. Thus with only four non-pictorial symbols, which were probably evolved from pictorial combinations, the early Egyptians could figure numbers up to 99,999. In later inscriptions, however, the symbolic methods gradually creep in. Thus *five* is represented by a five-rayed star, *six* by the star and a stroke, *seven* by the star and two strokes. This star may be meant to symbolise a spread-out hand, or it may have an astronomical reference to the 5 planets which, with the sun and moon, formed the seven divine luminaries. Less obvious symbols appear later for *seven*, *eight* and *nine*; and peculiar forms also seem to have been evolved for the various *tens*. Turning now to the Cuneiform inscriptions, we meet with a system very similar in its broad outlines to the early Egyptian Pictorial. The numbers up to *nine* are represented each by the proper number of the simple wedge-shaped character. *Ten* is symbolised by the angle-shaped character, two of which give 20, three 30, four 40, and five 50. *Sixty*, however, is represented by the same simple character as one, to which ten is added to make 70, two tens to make 80, and so on. Amongst the old Accadians this mode of numeration was continued throughout, numbers being written, and perhaps named, in the Sexagesimal scale. Thus the expression

   2,36,21

means $2 \times 60 \times 60 + 36 \times 60 + 21 = 9381$. Amongst the Assyrians, again, a distinct symbol, compounded of the unit and a small horizontal wedge following, was used for a hundred, and a prefixed ten gave the thousand. For example the Assyrians would write the above number

The Persians in their Cuneiform inscriptions seem to have banished all trace of the sexagesimal scale. Certainly the substitution in whole or in part of the denary for the sexagesimal scale marks an advance towards simplification. For many purposes, however, this seemingly awkward sexagesimal scale was really convenient; and not only did the Assyrians, and much later the Alexandrian astronomers, use it in the expression of fractions, but it survives to this day in the graduation of the circle and in the subdivisions of hours and minutes. Its origin was probably astronomical. The Accadians seem to have attained a high civilisation, and there is no doubt that in writing their numbers they had clearly grasped the idea of "place" as giving value to a sign. The Assyrian modification is from this point of view a retrogression, and not until we come to the cipher notation do we return to the scientific method of the ancient and almost mythical Accadians. These examples from the Hieroglyphic and Cuneiform modes of writing are for the lower numbers strictly Pictorial. For the expression of the higher numbers, the necessity for the Symbolic soon arose; while in the most ancient of all we have not merely the germ of "place-value," which is the peculiar pride of our cipher system, but the very thing itself.

In the old mathematical treatises of the Chinese another system of notation, largely pictorial, is met with. This notation is extremely cumbrous, and has all the appearance of having been invented in the first instance as a visual representation of the Abacus columns. Up to five, the numbers are represented by the requisite number of vertical strokes as in the other pictorial systems. *Six* is represented by a T-shaped character, and the higher numbers up to ten by the obvious addition of vertical strokes below the horizontal line, so that *eight* has the appearance of a set of wickets at cricket, with a fourth wicket laid across the tops instead of bails. *Ten* is figured by a horizontal stroke, with a circle or cipher at the right-hand end. The successive "teens" are obtained by replacing this circle by the proper digit symbol. Twenty is two horizontal strokes, one above the other; thirty, three—and so on to sixty, which is a horizontal stroke with a vertical drawn above it. Thus *sixty* is simply *six* turned upside down; while *eleven* is the same T-shaped symbol turned on its side. There can be little doubt, I think, that this system, with its convention between the meaning of isolated vertical and isolated horizontal strokes, or between six and sixty, seven and seventy, and so on, grew out of an attempt to depict in some convenient manner the indications of the Chinese form of abacus. The hundreds are a repetition simply of the units, the thousands of the tens, and so in alternation—any blank abacus rod being represented by the circle or cipher. Thus the number 527,068 is figured

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The conclusion that this is but an abacus product is borne out strongly by two considerations, namely, the particular mode of representing the 7, 8, 9 in the tens, thousands, hundreds of thousands denominations; and the manner of writing from left to right, so incompatible with the general tendency of Chinese writing. As will be seen later, this latter peculiarity will be adduced as evidence that the abacus was imported into China from the west. The use of the circle in a limited cipher significance is also of historic interest. There is no *à priori* reason for employing such a form of character to represent an empty rod on the abacus, so that it is hardly possible that two distinct races should have invented the same symbol. The probability rather is that the Chinese adopted this symbol from the Indians, among whom, according to Sir E. C. Bayley's researches, it developed from a symbolic form of ten.[8] It appears then that the Chinese pictorial system is rather a retrogression than a progression in the history of arithmetic, being a cumbrous and somewhat childish figuring of the abacus indications.

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[8] In some Chinese treatises a cross or X-shaped character is used for "four"—a form of character which is met with in the Bactrian and other old Indian numerals, where it has exactly the same significance. This is another strong argument for the Indian origin of the Chinese arithmetic.

We now pass to the symbolic stage, in which are included many widely diverse systems, their only common feature being the existence of distinct symbols for 20, 30, 40 and so on. The development of these various symbols is coëval with that of alphabets and syllabaries, and of civilisation generally. Nearly all the alphabets of the world have been traced through the early Phœnician to the final semi-alphabetic forms of the Egyptian Hieroglyphics. Even the numerous alphabets found all over Central and Southern Asia and in the Islands of the Eastern Archipelago, are believed to be descendants, through the old Magadhi alphabet of India, of the same great original. The numerals have certainly followed a similar course. Already in the Accadian Cuneiform and the Egyptian Hieroglyphics a few distinct symbols have crept in, invented obviously to save time in writing. With the growing need for more rapid writing, symbols continued to be invented, or perhaps more strictly, evolved from the original Pictorial representation. Now there is not the least doubt that there is a great conservative momentum in the mind of man. Even in these days of enlightenment and progress, the intensely practical Briton spells as if he knew not how he spoke; and the philosophic German says there are "three hundred five and sixty" days in the year! And the same mental habit of man is shown in his number-writing. The Accadians, by a wonderful generalization, had grasped the idea of "place-value"; but the system was necessarily cumbrous with *sixty* instead of *ten* as the notation unit. Their successors only partially took up the sexagesimal notation: and the idea of "place" was quite lost sight of with the introduction of the Decimal division of numbers. The Egyptians, again, never seem to have attained anything like the mathematical grasp of the Accadians; and, as a necessity, they found their solution of the problem, how to write numbers, in a multiplicity of symbols. A consideration of the Hieratic numerals will show clearly the nature of what is here called the Symbolic stage. (See Plate 1.) Not only are there distinct signs for the units up to nine, but the successive decades and hundreds are provided with peculiar symbols also. Sometimes one of a series is clearly a modification of one of its predecessors, as for example 20 of 10, 80 of 60, and the successive hundreds. This indicates one mode by which man invented his numerical symbols. The numerals of the Gupta Inscriptions and of the Maldive Islands, and the older Devanagari and the modern Cinghalese systems may be grouped, along with a number of ancient Indian systems, as of similar structure with the Hieratic. Several influences were at work in the formation of these symbolic systems, all of which seem traceable to the same ultimate source. One race would borrow from another, perhaps taking a symbol and applying it to a different number. Or the symbols might have phonetic values and be strung together to form a word or phrase of mnemonic value. Or the alphabetic or syllabaric symbol might be used which corresponds to the first letter or syllable of the name as spoken; as for example the Roman C. and M. With all these possible modes at the disposal of man, it is little wonder that in his inventive exuberance he should have evolved such a multiplicity of symbols. Whatever symbol became popular did so by a kind of natural selection. In making this selection, however, the hereditary tendency of man's mind was an important factor, and the principle of conservative momentum would certainly make itself felt, so that development would take place along the lines already laid down. At the same time, it would all be by way of simplification. To use initial letters when possible was a very obvious method, which we meet with in the early Greek and in the Ethiopic systems. From such a system would spring very naturally the idea of using the alphabet for the successive numbers—indeed there may have been a kind of mutual adjustment of the numeral series and the alphabet series. In this way the Hebrew and later Greek numerals became formed, and other alphabetic systems, such as the Georgian, Armenian, and older Turkish and Arabic. The Greeks borrowed their system from a people with a similar but fuller alphabet, as is shown by the fact that they had to throw in a special non-alphabetic symbol for six. Their special symbols for 90 and 900 were probably later introductions, to eke out the characters to the necessary twenty-seven. The Roman system, which is largely symbolic, is too well known to require special mention. Now of all these symbolic systems of numerals, the Greek alone was capable of being used for calculation. The thousands were the units repeated with a suffixed " dash "; but the calculator could omit the " dash " without fear of confusion. For the expression of higher numbers, octad and tetrad combinations were employed, much as we nowadays tick off our large numbers in groups of threes. Here the Greek came in contact with the principle of "place-value," but was still far behind the ancient Accadian.

The defect of the symbolic systems for calculating purposes was not, however, felt by their users; for they had the Abacus and like instruments, which were sufficient for their needs. The earliest form of abacus was a simple board covered with fine sand or dust. This surface was ruled into columns, which served for the different numerical denominations, the units, the tens, the hundreds, etc. In the columns thus made the numbers were marked by strokes or symbols. Latterly the sand was dispensed with, and pebbles (*calculi*) or counters were used on boards ruled into permanent columns. In another form, the counters were placed on lines as we place our "men" in backgammon. Still another form was the combination of rods and beads familiar to us all in the special modification of it called the *soroban*. If we could pass back to pre-abacus days, we should probably find our ancestors counting by bundles of ten or twenty, as savage races do now. From this mode of reckoning, the table of columns or abacus would be a very natural development. It has been already pointed out that the abacus has all the marks of great antiquity, so that its evolution is probably coëval with that of the numerals. Each numeral was essentially a shorthand expression for the idea or the name, and was a conception distinct from that of calculation. Hence it is little wonder that the numerals and abacus developed along perfectly distinct lines; and long before the numerals had passed from their symbolic stage amongst the early Indians or later Greeks, the abacus had attained its highest perfection. On the abacus the "place-value" of number was recognised—indeed could not be mistaken; and yet, if we except the Greek notation of tetrads and the cumbrous Chinese pictorial symbols, nothing corresponding to this had been evolved in numerals. In fact, so far, numerals were used as ideographs, not as arithmetical symbols.

The abacus may have had an influence in accelerating the transition to the next stage—a transition which seems to have taken place in India and in China only. This stage, which I have called the Decimal, is marked by the elimination of the special symbols of 20, 30, 40, etc., up to 90, which are henceforth written as "two-ten," "three-ten," four-ten," and so on.

The Chinese numerals give a very perfect example of the system, which is also found amongst the Tamils. They are shown in Plate I.[4] By whatever means and through whatever intermediate forms this great simplification was made, it signified a firmer grasp altogether of the nature of numbers. The symbol *ten*, in fact, is used in a new and quite conventional signification in these combinations. For example compare the 十三 (13) and 三十 (30). The former means ten (and) three, the latter three ten's, or *ten three-d*. In fact, in 三十 the 十 becomes a denomination rather than a number. The convention then for differentiating 十 in its two meanings is as follows: When 十 follows a number it is to be repeated that number of times; but when it precedes a number that number is to be added. Such a symbol as 十十 could mean either *two tens* (precisely as in the original pictorial method) or *ten ten-ned*, that is, 100. The Decimal method, however, stops here, and introduces a distinct symbol for 100 (namely 百), another for 1000, and so on. We shall give here the successive Chinese symbols, with the modern Japanese pronunciation of their names as being more familiar to our readers than the original Chinese pronunciation.

百	*hyaku*	=	100	or	10^2
千	*sen*	=	1000	or	10^3
萬	*man*	=	10,000	or	10^4

The next three stages in powers of ten are called *jū-man*, *hyaku-man*, *sen-man*, and are so written. Thereafter the new symbols go by ascents of 10,000.[5] They are as follows:

億	*oku*	(10^8)	溝	*ko*	(10^{32})
兆	*chō*	(10^{12})	澗	*kan*	(10^{36})
京	*kyō*	(10^{16})	正	*sei*	(10^{40})
垓	*gai*	(10^{20})	載	*sai*	(10^{44})
秭	*shi*	(10^{24})	極	*kyoku*	(10^{48})
穰	*jō*	(10^{28})			

[4] The Chinese symbols are of course written vertically, but for convenience in the text we shall write them from left to right.

[5] Such, at least, is the custom amongst the educated Japanese of the present era. The usual dictionary meanings of *oku* and *chō* are not the same as those given here; indeed the words seem to have been used more or less vaguely in former times in much the same way as Europeans use *billion*, *trillion*, etc. One authority gives three modes of progression; namely, ascent by *tens*, ascent by *ten-thousands*, ascent by *successive squarings*. The terms above *chō* are rarely used.

There are also terms for the decimal places as far as the 12th. These are:

分	*Bun*	10^{-1}	纖	*Sen*	10^{-7}
釐	*Rin*	10^{-2}	沙	*Sha*	10^{-8}
毫	*Mō*	10^{-3}	塵	*Jin*	10^{-9}
絲	*Shi*	10^{-4}	埃	*Ai*	10^{-10}
忽	*Kotsu*	10^{-5}	渺	*Byō*	10^{-11}
微	*Bi*	10^{-6}	漠	*Baku*	10^{-12}

The ideographs for these words have many of them very suggestive meanings. Thus the character for *Sen* means silk thread; for *Sha*, sand; for *Jin* and *Ai*, small dust; for *Byō* and *Baku*, hazy, cloudless aspect of the sky. Excepting the first three, however, which are common, these terms are rarely used outside the covers of a mathematical treatise.

The Tamil and Malayalam numerals follow closely the same course, and the process has to a certain extent appeared in the Cinghalese system, which may therefore be regarded as marking a stage during the simplification from the symbolic.

The peculiarities which distinguish the Decimal from our own Cipher or Indian system are apparent from the following comparative tables:

Chinese	一	四	十	十四	四十	四十一	四百四十一
Indian	1	4	10	14	40	41	441

The transition from the Decimal or Chinese system to the Cipher system is such an obvious one, especially with the Abacus columns in full view and in daily use, that our surprise is, not that the Indians of some 2,000 years ago should have made the step, but that the Chinese or Japanese should not. Here are two highly intelligent races possessed of a convenient arithmometre and of a system of number-writing which can be called a notation, and which in some respects approximates to the visual representation on their instrument. The one race brings the notation and Abacus into perfect accord, and begins the era of true science; the other makes no advance whatever, and even scorns to accept the perfected system, with which it has been face to face for centuries. It may be said, as M. Woepcke[6] said of the Greek mathematicians, that with the Abacus in hand, the Chinese and Japanese did not feel the want of a Cipher system. But if the theory is true that our Cipher system passed through the Decimal stage amongst a people who used the Abacus in a form exactly similar to the Chinese instrument, the argument ceases to have any great point. Woepcke's remark was made in his rather laboured attempt to explain why our numerals, given to us by the Arabs, who got them from the Indians, were not exactly the same as those used by the Arabs. A glance at the various Cipher systems figured on the Plate II. will show what variety of form has existed and still does exist amongst the nine digits. Either, then, the system grew up simultaneously in different districts which used their own peculiar modifications of the unit figures; or, the principle alone spread. Sir E. Clive Bayley has given good reason for the belief that the *cipher* is a modified *ten;* and as the *circle* and the *dot* are the only symbols in use as a cipher, however much the other figures may vary, it seems probable that the Cipher system was really developed in one district.

It is the cipher or *zero* which gives the system its peculiar power. Both words are from the same Arabic origin (sifr), which is simply the translation of the Sanskrit word "sunya," which means emptiness. This in fact was one of the names applied to the empty column or rod on the Abacus, and meant merely a condition or state. The Chinese similarly use the word *Ling* (零), pronounced *Rei* by the Japanese.[7] When a Japanese is reading out a series of numbers

[6] Journal Asiatique, Series 6, Tom I.

[7] The primary meaning of 零 is 'the last drops of a shower,' or 'slow rain,' hence generally 'remainder,' 'residuum,' 'fraction,' etc. The meaning of 'zero' is generally supposed to come from these, as being of the nature of a degenerate number, something so small as to be valueless. Such an asymptotic derivation, as it might be called, seems almost too mathematical to be satisfactory. I should suggest an Abacus derivation as being at least as plausible. That is, just as our *cipher* and *zero* can be traced back to the Arabic *sifr* which was applied to the "empty" abacus rod or column; so may the Chinese 零 have been applied to the abacus rod from which the last counter had been made to

to the *Soroban* worker, he inserts the *rei* where no significant figures occur. Thus instead of saying simply *san sen go* (3005), as he would in ordinary conversation, he reads *san sen rei rei go*. Formerly the pure Japanese word *tonde* (skipping) was used for the same purpose. The symbol (O) representing *rei* in mathematical works has been already referred to. In these days it is used in bank-notes and bank-books, exactly as our cipher is, for spacing out the numbers, the symbols *jū, hyaku, sen, man* being omitted. It never has been used, however, as a *cipher of calculation;* and it bears the evidence, as already pointed out, of being originally an importation from the west into China.

Taking the Chinese numerals as the type of the Decimal Stage, and our own numerals to represent the Abacus columns, we might imagine the development to the cipher stage as taking place in this wise. Up to *nine*, both Abacus and numerals are in accord. At *ten*, however, the next rod of the Abacus is brought into requisition, so that *ten* is represented by a combination of a *one* and a *void*, which has no similarity to the single symbol 十. Up to nineteen, however, there is similarity, the 十 of 十九 being comparable to the 1 and the 九 corresponding to the 9 on the two contiguous abacus rods. The decade numbers 二十, 三十, 四十, etc., correspond very well with the Abacus indications 20, 30, 40, etc., where now of course the 十 is comparable to the empty space. The similarity somewhat breaks down at 二十一 (21); but the approximate similarity would suggest dropping the 十 here and writing 二一.[*] By this simplification, which could lead to no confusion, the 十 only appears in the twenty, thirty, etc., since there it is required to denominate the two, three, etc. But consistency would suggest to write 一十 for *ten* or *one-ten*, exactly

"drop." The arguments in favour of this derivation are these: the term in its zero significance is originally arithmetical; arithmetic was formerly inseparable from the abacus; and we have in our own *cipher* an analogous derivation.

[*] This in fact is done by the Japanese in marking their counters in the game of *go* (棊), the 十 being dropped to save room; so that 34, 68 are written 三四 六八. The same system of contraction is not continued to the hundreds, however, a modified symbol (ㄗ) for *Hyaku* being introduced. Postage stamps and coins are similarly marked.

as 二十 stands for *two-ten*; and the "teens" would then stand a good chance of being treated like the twentys and thirtys. This mode of writing and saying *ten* is indeed met with in Japanese literature. Thus everything would come into accord with the Abacus representation, and the symbol 十 appearing only as a denomination would cease to be called *ten* and be named anew by the Abacus name for empty space. The extension of the system to higher numbers and the vanishing forever of symbols for the successive powers of *ten* would be an obvious improvement. In some such manner then—and Sir E. Clive Bayley has given historical evidence in support of the theory—did the Decimal pass into the Cipher Stage of numbering. The natural tendency of the human mind to simplification, aided at the right moment by the indications of the Abacus, produced from a chaos of symbols a numerical system which has determined more than any other one thing the rise and progress of mathematical science. In the history of Arithmetic the only event which is at all worthy to be compared with the introduction of the cipher is the discovery of Logarithms.

The spread of the Cipher system into Europe is itself an event of deep historical importance. Of all other systems of numbering, the Greek alone possessed any flexibility as a medium for calculation: but its operations were no doubt largely aided by the Abacus. The Sexagesimal modification, perpetuated if not introduced by Ptolemy and the Alexandrian School, was a significant improvement and especially available for astronomical calculations. Since in this Sexagesimal system ξ (60) was the last symbol needed, the next symbol, o, has been supposed to have been the origin of our cipher. The Neo-Pythagoreans certainly used such a symbol and used it in a partial cipher signification; but there is no evidence that they knew of the decimal cipher previous to the 7th or 8th century. Whereas there is evidence in the writings of Aryabhata (360 A. D.) that the Indians knew the principle of "place-value" and used the zero at that time.[*]

The question we have now before us is: What causes prevented the development of a Cipher system in China or Japan? A partial explanation may be found in the mode of writing. The Chinese write in vertical columns from above downwards; and if they ever are

[*] See Sir E. Clive Bayley's Second Paper for a full discussion of this point.

compelled to write in a horizontal line they work from right to left. Now the Abacus is worked from left to right, a fact which tends to prove incidentally that the Abacus is not indigenous to China. The similarity between the numerals as written and the Abacus indications of the same would not be so striking to the Chinaman as to the Aryan or Semite, since these wrote in horizontal lines. Now so far as evidence goes, our numeral systems all passed to the races of Aryan origin through the Semitic peoples, who generally wrote from right to left. As will be seen in the subsequent part of the present paper, the Semite named his numbers by beginning with the unit or smallest denomination. Thus in Arabic it is five and twenty and one hundred, instead of one hundred and twenty-five. But in writing down a number he would write it as he named it; and as he both, so to speak, wrote and named backwards, the result would appear as it is on the Abacus, 125. Now the early Indian spoke like the Arab, but wrote from left to right; while the Chinese always spoke as we do now but tended to write from right to left. Hence if the Abacus had been an Indian or Chinese invention, the columns would probably have gone the reverse way, with the units to the left, so that one hundred and twenty-five would have appeared as 521. This argument of course cannot be urged in the face of evidence to the contrary; for we know that in ancient days both modes of writing were in use by the same people. In some inscriptions indeed the writer has turned backward along the next line, ploughman-like. Still as the Chinese write in vertical columns, so the Semitic peoples generally write from right to left and the Aryan from left to right. Hence, unless there were definite evidence to the contrary, we should be inclined to regard the Abacus as not being primarily an Aryan invention, but more probably introduced to the Aryan races through the Semitic peoples. And this in itself is not improbable, inasmuch as the Semites were the great commercial peoples of the ancient world. There is one consideration which prevents us regarding it as a Semitic Invention, namely, the lack of the inventive faculty in the Semitic mind. And yet such a natural development of the early finger exercises as the Abacus is, might well lead to its invention even by a much less civilised community. In any case, we must regard the rise of commerce as an important influence in the evolution of all forms of calculating boards.

This diversity in the mode of writing and mode of placing on the Abacus a given number is hardly a satisfactory explanation of the persistency of the instrument amongst the Chinese; for the Tamils, who write from left to right and who have lived in close contact with cipher-using peoples, use to this day a system of numerals exactly similar to the Chinese. It remains to enquire as to the existence of some mental or linguistic peculiarity possessed by the Tamils and Chinese and not possessed by Aryan races. In other words—for we all believe in the doctrine of the survival of the fittest—are there any linguistic or mental peculiarities which may make the Abacus more efficient, that is, more rapid and more certain, than ciphering?

There is not the least doubt that as used by the Japanese the Abacus is for ordinary arithmetical operations more efficient than figuring. This efficiency I think is traceable to their peculiarly suitable mode of numeration or number-naming. At first sight, many would be inclined to think there was no essential difference between the Japanese or Chinese numeration system and our own. But a closer study reveals to us a very striking difference indeed, which it is now our object to discuss.

The question of the nomenclature or naming of numbers opens up another and quite distinct line of enquiry; and Comparative Numeration, as it might be called, may lead to a clearer understanding of the historic bearing of the Abacus. Here we come face to face with one of the deepest problems of philology, the origin of the names of numbers. So far as regards the Aryan family of languages, some small advance seems to have been made towards the solution of this problem. Thus "three" has been connected with the root meaning to *pass over;* "seven" with the root meaning to *follow;* "nine" with the Sanskrit pronominal base meaning *new*. That is, to quote Sayce, three is named from its excess, seven from its following the foregoing numbers, while nine is the new number. The naming of three from its excess has received an ingenious explanation by Dr. Koelle,[10] who connects it with the length of the middle finger.[11] Reckoning with the aid of

[10] On the origin of the Turkish Numerals, Journal R. A. S., xvi (1884).

[11] The general theory that the names of the numerals in all languages are connected with the peculiarities of the hand is as highly probable as it is difficult

the fingers is of course the most natural of all methods and is the source of our wide-spread decimal system. In some savage tribes of the present day the very names used for *five* and *ten* signify "one hand" and "two hands;" and this metaphorical way of speaking is carried on by means of the toes, so that *twenty* is called "one man." Using the toes as well as the fingers seems to have been quite a

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of proof. The simple finger-theory, as it might be termed, although it may hold for some few tongues, in general breaks down very early in the series of numbers. Before any such theory can be profitably discussed, it is necessary to know the natural order in which a given race uses the fingers in counting. That considerable diversity exists amongst peoples in this respect may be shown by the following examples. A European, in "telling" off his fingers numerically, would probably begin with the thumb of his left hand, marking each finger in succession by contact with the fore-finger of his right hand. He might then pass to his right hand to complete the ten, or simply repeat the operation on the left hand. An English school girl, who usually counts by a kind of five-fingered exercise on the table or desk beside her, first raises the hand slightly above the surface and then, beginning with the little finger, brings down each finger-tip in succession until 5 is counted, after which a fresh start is made with the little finger. Thus the middle finger always means 3 or 8, the fore-finger 4 and 9, and so on. The North American Indians always begin with the little finger of the left hand and finish with the little finger of the right hand. According to Dr. Koelle, the Turks and the inhabitants of Western Africa begin, like the North American Indians, with the little finger of the left hand, but, unlike them, end with the thumb of the right hand. The Japanese, again, use only one hand after a fashion which seems to be peculiar to them. Beginning with the left hand open, they turn the thumb in towards the palm to represent *one*, bring down the fore-finger over it for *two*, and so on in succession till *five* is reached with the closed fist. For *six*, the little finger is raised again, and one by one the preceding operation is undone till ten is reached with the open hand. Thus the little finger up alone means either 4 or 6; up along with its fellow, 3 or 7; all the four fingers up, 1 or 9. The Japanese have also several peculiar methods of silent bargaining, in which the buyer and seller grip each other's hands. In one of the most common of these, the price is indicated by the number of fingers grasped, the little finger meaning one, the thumb alone meaning five. Thus *three* is indicated by the little, ring, and middle fingers; *eight* by the thumb, fore, middle and ring fingers. Ten may be shown by grasping the second joint only of the thumb. The nature of the bargain sufficiently determines the money unit employed, or the possible range of the bargaining. If it is necessary to indicate two denominations of money, the higher is separated from the lower by a grasping of the wrist.

favourite mode of numeration, as is evidenced by the existence of numerations which ascend by *twenty*;—the odd decades (thirty, fifty, etc.) being words compounded of ten and the preceding decade. This method is found amongst tribes of the Caucasus and Hindu Khush, and in such widely scattered communities as the Basque, the Ainu, and the Mexican. The French names *soixante-dix*, *quatre-vingt*, *quatre-vingt-dix*, which have nearly displaced the regularly formed *septante*, *octante*, *nonante* still used in Switzerland and in the South of France, are perhaps a revival in spirit of the same method lingering through centuries. The tendency shown in some languages to group numbers in fours and sixes is not so easily explained, though the four-fold method may probably be referred to the fingers, as distinguished from the thumb. The grouping in sixes and twelves again, I believe to spring partly from the sacredness of the number *three*, which has its origin far back in the days of the dawn of reason. Everything tends to show that, as man developed socially, *duality*, as a quality to be expressed by language, preceded *plurality*; and the co-existence of dual and plural inflexions marks a stage in the growth of the human mind which the higher races of the present day have far outstripped. It is in peoples of low intellectual power that we find a fulness of explicitly expressed meaning that is unnecessary in the race of higher mental grip. The probability is then that the naming of the number *two* long preceded the naming of *three*, which, as in low savage races of historic times, would originally be synonymous with many. Hence the passing to three as a distinct conception would be a great stride in the mental progress of man, and might well perpetuate itself in a kind of superstitious reverence, especially in the presence of the three great natural divisions of sea, earth, and sky. Then again to early man, when writing was unknown, the use of a number which could be halved and "thirded" and quartered would be very natural—only too natural indeed as we know to our mental confusion now. It was this apparent simplicity, real then of course, which resulted in the evolution of our complex European tables of weights and measures. And the existence of such complications is, I think, a proof by the way, that the Abacus, with its strongly marked decimal character, never attained in Europe anything like the flexibility in calculation which it has attained in the

East. In any case, however, the popularity of twelve as a basis for reckoning may be reasonably traced to its possessing many simple submultiples, and three amongst others. Duodecimal scales of numeration have been found amongst savage peoples, notwithstanding their ten fingers; and we may safely assert, that had man possessed six fingers, the decimal scale would never have been mentioned outside mathematical treatises.

After all, however, "ten" has been the favourite numeration unit; so much so indeed that such numbers as eight and nine have been sometimes named in terms of it by a backward process very similar to the manner in which the Romans write IX for 9, XL for 40, and so on. Thus, amongst the Dravidian peoples, *nine* is usually expressed as *one-ten;* and in Finnish and some related languages *eight* is expressed as *two-ten*. The same method is quite usual in the higher decades even among Aryan peoples, as for example in the Latin *duo-de-viginti*, *un-de-viginti*.

Passing now to the second decade of numbers, we notice that these have almost universally been named by combining or modifying the names of the first ten. Thus the derivation of the English *eleven* and French *onze* is simply *one-ten*, and of the English *twenty* and French *vingt*, *two-tens*. Twelve and twenty indeed have the same derivation, just as the very obvious Japanese *jū ni* and *ni jū*. In some languages 'twenty' is a distinct word having no apparent philological relationship to ten; and in Turkish, invention of terms is carried up to *fifty*, *sixty* being the first decade number which bears *six* on its face. This I regard as showing that the Turks possessed comparatively feeble powers of generalization, as being in fact a race mentally inferior in this respect to the Semitic peoples. In cases, however, in which the name of the successive decades were formed from the lower numbers, it was not always as in the Aryan and Chinese languages. The Hebrew 'twenty' was the plural, originally the dual, for 'ten'; and the succeeding decade names up to a hundred were the plurals of the corresponding digits, threes, fours, fives, etc.

Generally, and especially in the inflexional languages, the principle which philologists call Phonetic Decay has been very busy with the names of the higher numerals. This is shown especially in our own called "three (and) twenty," "five (and) sixty;" and in English this combination is often still employed in conversation. The influence of the notation has however compelled the more practical time-saving mind of the Briton to shake himself free of the old method in naming numbers above twenty; but hereditary habit is too strong to allow him to alter his "teens." The Romance languages largely follow their common source, which as we all know had latterly at all events "twenty-three, sixty-five," etc. The Greeks and Latins indeed, seem, like ourselves, to have adjusted their nomenclature in the higher decades to suit the direct way of reading the inverse notation borrowed from the East. The Sanskrit, however, resisted this harmonising all through even up to the highest named numbers. Thus 325 is named, "Five and twenty and three hundred," exactly as in Arabic and in Early Hebrew. Hence Hindustani, one of the modern representatives of Sanskrit, which uses a modification of the Arabic in writing, is thoroughly consistent in notation and nomenclature; but all the other Gaurian languages of India are saying one thing and writing another. If we may judge from the early Sanskrit writings on mathematics, the Abacus indications seem to have been read backwards, a most un-Aryan like procedure, and strongly suggestive of the remark made above, that the Abacus was borrowed by them from some neighbouring peoples. In the Keltic group of languages the same method of number-naming is adopted all through the decades; and in Welsh, Gaelic and Irish, the process is complicated by inserting the noun in the middle of the number. Thus eighteen men, twenty-six sheep, are expressed "eight men ten," "six sheep (and) twenty." In fact the older the dialect, or the less influenced it has been by contact with non-Aryan peoples, the more clearly marked is the inverse mode of naming numbers among the Aryans and the method has survived in the expression of the "teens" in almost all languages down to the present day.

Now Chinese and Japanese[15] are as direct as they can be in their number-naming, passing invariably from the general to the special, from the larger to the smaller. This fact, which I believe affords the explanation we are in search of, at once suggested to me the advis-

[15] This statement applies to the *original* Japanese numerals as well to those of Chinese origin.

ability of searching other languages for their systems of numeration. Numerals are such an important element in all philological research, that this might seem at first sight a very simple operation. But here in Japan, where there is no library for general reference, I have found it no easy matter; and very frequently the list of numerals obtained just skipped from ten to twenty, as if the intermediate ones were of no account. Thus, in the long list of Turanian numerals given at the end of Bunsen's Philosophy of Universal History, comparatively few have the names for 11 and 12; hence for much of the information obtained I have to thank my linguistic friends in Tōkyō, and especially the Vice President and the Corresponding Secretary of the Society for the trouble they have taken in ferreting out the facts required.

The general facts of the investigation are these: The Aryan and Semitic peoples, almost without exception, name the smaller number first,—thirteen, fourteen, and so on. The Ural-Altaic, the Dravidian, the Tibeto-Burman and the Chinese peoples, with as rare exceptions, name the larger number first,—ten-three, ten-four, etc. The following two lists give all the languages that have been investigated, with the exceptions added.

I. Inverse Method:—Smaller component first.—Aryan Languages; Assyro-Babylonian, Sabean, Hebrew, Syriac, Arabic, and probably Semitic generally; Shina (Hindu Khush tribe); Ainu; Malay, Malagasi; Yoruba; Apache, Navajo; Maya (Ancient Mexican).

Exceptions:—Modern Greek, Roumanian, Ethiopic.

II. Direct Method:—Larger component first.—Chinese; Korean; Japanese, Manchu, Samoied, Turko-Tatar, and Siberian generally, Magyar; Burmese, Tibetan, Lepcha, Singpho, Changlo, Mikir, Miri, Kunâwari, Dophla, Naga, Shendu; Siamese, Miautsi; Avâr; Dravidian languages, Tamil, etc.; Kolarian languages, Ho, Savara, etc.; Alarodian languages, Lezian, etc.; Nubian dialects; Vei; Hottentot; Hausa; Coptic; Basque; dialects of Hindu Khush tribes, Khowar, etc.; and many languages of the North American Indians.

Broadly then we may say that, excepting the great Ayran and Semitic families, the Malay group, and some of the languages of Central America, New Mexico and Western Africa, all mankind tend to numerate in what we have called the Direct method.

good illustration of the principle. It may be, then, that the Inverse mode of naming compound numbers betokens a stronger individuality, a more pronounced determination on the speaker's part to be understood, but that it has in large measure been replaced by the direct mode under the powerful influence of a borrowed notation.

Summary and Conclusion.

The Abacus, as used in China and Japan, bears, on the very face of it, evidence of a foreign origin. The numbers are set down on it with the larger denomination to the left, a result which could come from a people either speaking and writing inversely, or speaking and writing directly. Historically, the home of the Abacus is in India; but it could hardly have been invented by the Aryan Indians, who wrote directly and spoke inversely. The probability is they borrowed it from Semitic peoples, who were the traders of the ancient world; and these may have invented it, or, as is perhaps more probable, received it from a direct-speaking, direct-writing race, such as we know the highly cultured Accadians to have been.

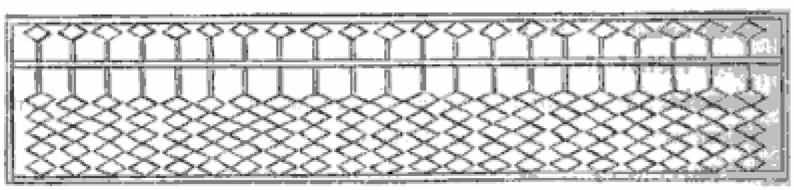
In early times the Abacus, as being an evolution from the natural Abacus—the human hand—pursued a course of development entirely different from that of the graphic representation of numbers. This latter we can trace through four stages,—the Pictorial, the Symbolic, the Decimal and the Cipher. The Pictorial we find in the Egyptian hieroglyphics, the Accadian Cuneiform, and the technical Chinese of mathematical treatises; the Symbolic in the numerous methods which grew up with the development of alphabets and syllabaries; and the Decimal in the simplifications of these, which live to-day in the Chinese and Tamilic systems. Once the Decimal stage was reached, its general similarity to the Abacus indications suggested bringing them into still closer correspondence.

This advance seems to have taken place amongst the Aryan Indians, who along with the Aryans of the West very soon discarded the Abacus for the more convenient Cipher notation. With the Chinese, Tamils and Malayalams of South India, no advance was made in this direction; the reason being simply that the Abacus better suited their numeration. These peoples speak directly, so that their nomenclature fits in perfectly with the Abacus indications, and makes its manipulation more rapid and certain than calculation by ciphering. An Aryan Indian with his inverse speaking could never work the Abacus with the same facility as a Japanese unless he worked from right to left—a mode of procedure quite foreign to his nature. It is not so foreign to Chinese and Japanese, however, to work from left to right, as each individual character is formed in this way. It may be safely concluded that only amongst a people who used the direct mode of naming numbers, or who with the inverse mode of naming preferred the inverse mode of manipulating, could the Abacus in the form in which it was evolved ever attain the beauty of action of the Japanese *Soroban*. To the discussion of its peculiar merits we now proceed. We shall employ throughout the Japanese name, which it should be noted is simply a mispronunciation of the Chinese name—*Swanpan*.

PART II.—THE SCIENTIFIC ASPECT.

The Soroban may be defined as an arrangement of movable beads, which slip along fixed rods and indicate by their configuration some definite numerical quantity. Its most familiar form is as follows. A shallow rectangular box or framework is divided longitudinally by a narrow ridge into two compartments, of which one is roughly some three or four times larger than the other. Cylindrical rods placed at equal intervals apart pass through the ridge near its upper edge, and are fixed firmly into the bounding sides of the framework. On these rods the counters are 'beaded.' The size of the counters determines the interval between the rods, the number of which will of course vary with the length of the framework. Each counter (Japanese *tama*, or ball) is radially symmetrical with respect to its rod, on which it slides easily. Looked at from in front of the box, the form in perspective is that of a rhombus, the rod passing through the blunt angles. This double cone form makes manipulation rapid, the finger easily catching the ridge-like girth of the *tama*. On each rod there are six (sometimes seven) *tama*. Five of these slide on the longer segment of the rod, the remaining one (or two) on the shorter. When the *tama* on any segment of a rod are set in close contact, a part of the rod is left bare. The length of this bare portion is determined by a double consideration. It must be long enough to be clearly visible, and yet not so long as to make the action of the fingers irksome by reason of excessive stretching.

When a Soroban is lifted indiscriminately, the counters will take some irregular configuration upon their rods, being limited in their motions by the bounding walls and the dividing ridge. To prepare it for use, the framework is tilted slightly with the smaller compartment uppermost, so that each set of five counters slips down to the bounding wall end of its rod and each single counter[16] on its short rod slips down upon the upper surface of the dividing ridge. The framework is then gently adjusted till all the rods become horizontal, so that if any counter is shifted it will have no tendency to move back to its former position. By a sweep of the finger tips along the surfaces of the single counters, these are driven from their contact with the dividing ridge to the other extremities of the rods. In this configuration in which the counters are all as far away as possible from the dividing ridge, the Soroban is prepared for action. The number represented is zero. This position is shown in Fig. 1.



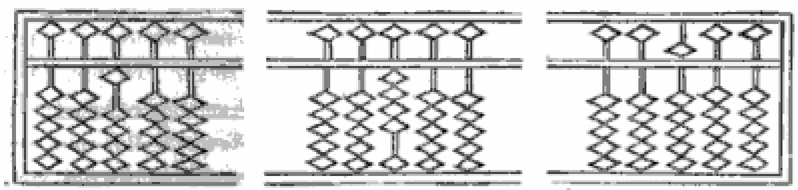
(Fig. 1.)

Let now any first counter of a set of five be moved till it is stopped by the ridge, as shown in the first diagram of Fig. 2. This will represent 1, 10, 100, 1000, etc., as may be desired. Let it represent 1, then a second moved up will give us 2, a third 3, a fourth 4. This

[16] We shall henceforth only speak of *one* counter as being on the short rod. The two counters, although facilitating somewhat certain operations in division, are not really necessary, and their use is exceptional.

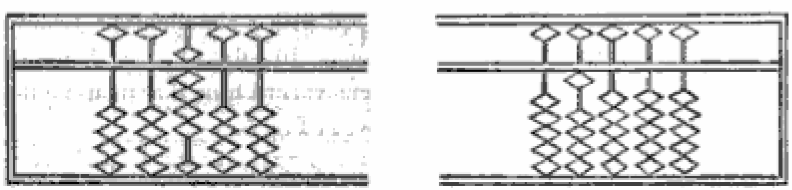
last is shown in the second diagram of Fig. 2. The last moved up will of course give 5; but this number is also given by pushing back the five counters to their zero position and bringing down the corresponding single counter to the ridge. This is shown in the last diagram of Fig. 2.

I. II. III.



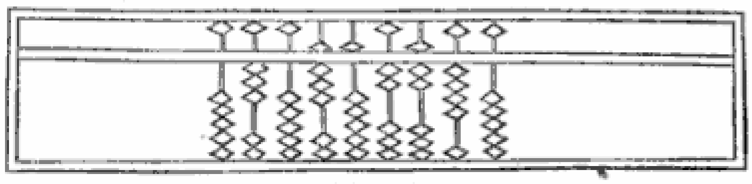
(Fig. 2.)

Leaving this single one in position, we get 6 by pushing up 1, 7 by pushing up 2, and so on till 9 is reached as shown in Fig. 3. The number 10 is then represented either by moving up the last counter, or more usually by clearing the rod of all its counters and moving one up on the next rod to the left, as shown also in Fig. 3.



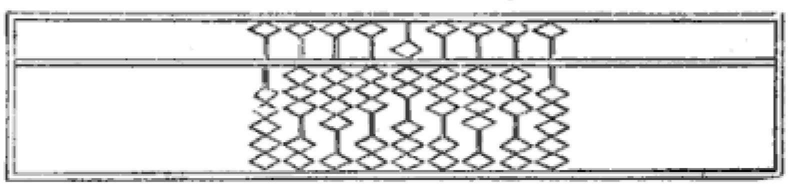
(Fig. 3.)

The mode of representing any number is thus obvious, being simply a mechanical model of our cipher system. Each rod corresponds to a definite figure 'place' (Japanese *Kurai* 位) or power of ten. One being first chosen as the unit, the next to the left is the 'tens,' the next the 'hundreds,' the next the 'thousands' and so on; while the successive rods to the right will represent the successive decimal places—tenths, hundredths, thousandths, etc. When the counters are as far as possible from the dividing ridge they have no value; when they are pushed as *near* the ridge as possible they have values as already indicated. The single counter when pushed down upon the ridge has five times the value of any other counter upon that rod. In Fig. 4, the number 3085·274 is shown. The mark V is placed over the 'units' rod.



(Fig. 4.)

The operations of addition and subtraction are self-evident. Thus let it be required to add to this number 352·069. On the 'hundreds' rod push up 3; and proceed throughout whenever it can be done in this way. On the 'tens' rod, however, where only two counters are left, it is impossible to push up 5. But since 50=100—50, the addition is effected by pushing up one counter on the 'hundreds' and removing 5 from the 'tens' rod. This gives of course 4 on the 'hundreds' rod and leaves 3 on the 'tens.' Then push up 2 on the 'units' rod; then 1 on the 'tenths' rod with a simultaneous removal of 4 from the 'hundredths' rod, since 10—6=4; then 1 on the 'hundredths' rod with a simultaneous removal of 1 from the 'thousandths' rod. The final result 3437·343 is given in Fig. 5.



(Fig. 5)

Subtraction is executed in a similar manner. It will be noticed that these operations involve no mental labour beyond that of remembering the complementary number, that is, the number which with the given number makes up 10. A glance at the configuration on any rod is sufficient to show if the addition (or subtraction) of a named number can be effected on it; and if this cannot be, it is necessary simply to add (or subtract) one to (or from) the next higher place and subtract (or add) the complementary number from (or to) the place in question. In first experimenting with the Soroban, an operator who is accustomed only to our Western modes of figuring is apt to add mentally, and then set down the result on the instrument. Such a mode is inferior of course to the ordinary figuring method, being liable to error, inasmuch as the number that is being added is not visible to the eye at any time, and the number that it is being added to disappears in the operation. But if any one will take the trouble to dispossess himself of his Western methods and work in the manner indicated, he will find Soroban addition and subtraction both more rapid and more certain, because attended by less mental exertion, than in figuring. The one seeming disadvantage in the Soroban is that the final result of each step alone appears, so that if any error is made, the whole operation must be carried through from the beginning again. Almost all writers on China or Japan, who have noticed the instrument, bring this forward as a serious disadvantage. But such a conclusion is a hasty one, and shows the writer to possess but small acquaintance with Soroban methods, and little regard to the true aim of calculation. For after all it is the result we wish; and if an error has been made, repetition is necessary both with Soroban and ciphering. The mean position of an accidental error is of course half-way through; and this would tell in favour of the ciphering system. But on the other hand, the Soroban is, on the average much more rapid than ciphering, and less liable to error. Only a lengthened series of comparative experiments could establish whether there is any real disadvantage at all.

Multiplication.

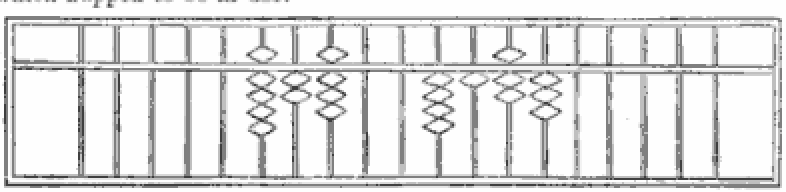
Multiplication on the Soroban differs but slightly from our own methods, being effected by means of a Multiplication Table—*ku ku gō sū* (九九合数),[12] literally, nine-nine combining number. Two peculiarities distinguish this table from ours. First, there is a complete lack of interpolated words like our "times," the multiplier, multiplicand, and product being mentioned in unbroken succession; and second, the multiplier, that is the first named number, is always the smaller. Thus the multiplication table for six runs:

[12] Generally called simply *ku ku*.

ichi	roku	roku
ni	roku	jū ni
san	roku	jū hachi
shi	roku	ni jū shi
go	roku	san jū
roku	roku	san jū roku
roku	shichi	shi jū ni
roku	hachi	shi jū hachi
roku	ku	go jū shi

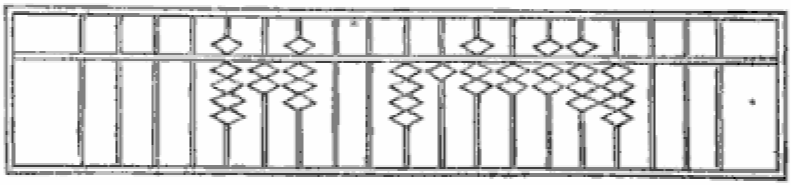
It is unnecessary to go to 12 as we do. Knowledge of a multiplication table for any number higher than 9 would retard Soroban manipulation. We British at least are compelled to learn up to 12 because of our monetary system; and it is often serviceable to know the table for 16. One is early struck by the inability of most Japanese students to multiply by 12 or even 11 in one line.

In multiplying two numbers together on the Soroban, the operator sets the two numbers somewhat apart on the instrument, the multiplier being to the left, the multiplicand to the right. There must be left to the right of the multiplicand a sufficient number of empty rods, a number at least equal to the number of places in the multiplier. The operation is essentially the same as ours; only instead of multiplying the multiplicand by each figure of the multiplier as we do, the Japanese multiplies the multiplier by each figure of the multiplicand. As the operation goes on the multiplicand gradually disappears, so that finally only the multiplier and product are left on the board. An example will render the method clear. Let it be required to multiply 4143 by 928. Set these on the Soroban, the multiplier anywhere to the left, and 3 empty rods at least to the right of the multiplicand. Henceforward in the diagrams we shall represent visually only the counters which happen to be in use.



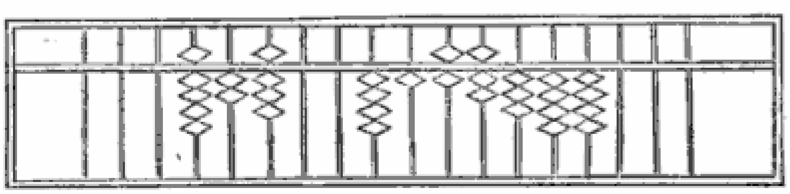
(Fig. 6.)

Multiply 8 by 3 and set 24 on the Soroban so that the 4 lies just as many places to the right of the multiplicand 3 as there are figures in the multiplier. This 4 is of course in the 'units' place of the product; and we shall continue to name the other places accordingly. Next multiply the 2 by 3, and add the product 6 to the 'tens' rod. This gives us the result so far 84. Lastly multiply 9 by 3. This requires 7 to be added to the 'hundreds' rod, and 2 to the 'thousands' rod. But before this latter operation can be done, the 'thousands' rod must be cleared of its multiplicand 3, which having completely served its purpose may easily be removed, and indeed is better away. Since 3 is to be removed and 2 added, it is sufficient to remove 1 and leave 2. The result so far is shown in Fig. 7.



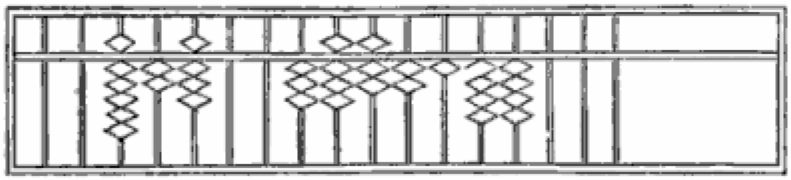
(Fig. 7.)

Now proceed to multiply with the next figure of the multiplicand, 7 namely:—$7\times8=56$, of which the 5 is to be added to the 'hundreds,' and 6 to the 'tens' rod; $7\times2=14$, that is, 1 to the 'thousands,' 4 to the 'hundreds;' $7\times9=63$, that is, leave 6 on the 'ten thousands' rod by taking off 1 from the 7 and add 3 to the thousands. The result of this operation is given is Fig. 8.



(Fig. 8.)

The operations with 1 and 4 are similarly carried out, care being taken to add the numbers which make up each several product in their proper places, and to suppress the multiplicand figure at the final operation with the same. The final result is given in Fig. 9.



(Fig. 9.)

It will be noticed that in all addition or subtraction processes, the number is added to or taken from the rod rather than from the number on the rod. The eye can tell at a glance if this operation can be effected on the rod in question, or if the next rod to the left has to be called into play. Mental labour is thus reduced to a minimum. The operator hears or utters a certain sound, which means one of two operations. A glance shows which of these it must be; and the fingers execute a certain mechanical movement which accompanies the sound of the words as naturally as the fingers of a pianist obey the graphic commands of a Sonata.

We see then how well fitted for Soroban use is the Chinese and Japanese momenclature of the numerals; and how ill adapted all such systems must be which say sixteen and five-and-twenty instead of teen-six and twenty-five.

Division.

Division on the Soroban, although essentially the same as our own Long Division, is in many respects peculiar and almost fascinating. The *art* of it is based upon a Division Table, called the *ku ki hō* (九歸法) or Nine Returning Method, which is learned off by heart. This we give in full as it is pronounced, with an accompanying translation as literal as possible.

Division Table for *Ichi* (one).

ichi is shin ga in jū	one one gives one ten
" ni " " ni "	one two " two tens
" san " " san "	" three " three "

and so on to

ichi ku shin ga ku jū	one nine gives nine

Division Table for *Ni* (two).

ni ichi ten saku no go	two one replace by five
" ni shin ga in jū	" two gives one ten
" shi " " ni jū	" four " two tens
" roku " " san jū	" six " three "
" has " " shi jū	" eight " four "

This Table could well stop at "ni ni shin ga in jū", since the higher ones are simply combinations of the first two. This is recognised by the absence of the "two five" statement.

Division Table for *San* (three).

san ichi san jū no ichi	three one thirty-one
" ni roku " " ni	" two sixty-two
" san shin ga in jū	" three gives one ten

The rest is obvious, being indeed but a repetition of the first three statements.

Division Table for *Shi* (four).

shi ichi ni jū no ni	four one twenty-two
" ni ten saku no go	" two replace by five
" san shichi jū no ni	" three seventy-two
" shi shin ga in jū	" four gives one ten

Division Table for *Go* (five).

go ichi ka no ichi	five one add one
" ni " " ni	" two " two
" san " " san	" three " three
" shi " " shi	" four " four.
" go shin ga ni jū	" five gives one ten

Division Table for *Roku* (six).

roku ichi ka ka no shi	six one below add four
" ni san jū no ni	" two thirty-two
" san ten saku no go	" three replace by five
" shi roku jū no ni	" four sixty-four
" go hachi jū no ni	" five eighty-two
" roku shin ga in jū	" six gives one ten

Division Table for *Shichi* (seven).

shichi ichi ka ka no san	seven one below add three
" ni " " " roku	" two " " six
" san shi jū no ni	" three forty-two
" shi go jū no go	" four fifty-five
" go shichi jū no ichi	" five seventy-one
" roku hachi jū no shi	" six eighty-four
" shichi shin ga in jū	" seven gives one ten

Division Table for *Hachi* (eight).

hachi ichi ka ka no ni	eight one below add two
" ni " " " shi	" two " " four
" san " " " roku	" three " " six
" shi ten saku no go	" four replace by five
" go roku jū no ni	" five sixty-two
" roku shichi jū no shi	" six seventy-four
" shichi hachi jū no roku	" seven eight-six
" hachi shin ga in jū	" eight gives one ten

Division Table for *Ku* (nine).

ku ichi ka ka no ichi	nine one below add one
" ni " " " ni	" two " " two
" san " " " san	" three " " three

and so on to

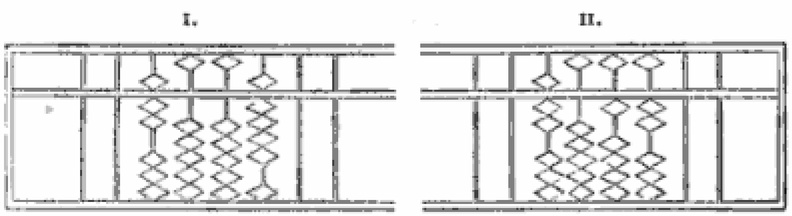
ku hachi ka ka no hachi	nine eight below add eight
" ku shin ga in jū	" nine gives one ten

It will be noticed that the essential parts of the division tables take no account of the division of a number higher than the divisor. Hence in division, the larger number is named first; whereas in multiplication, as we saw above, the smaller number is named first. Thus the Japanese gets rid of such interpolated words as "times" and "into" or "out of," which are necessary parts of our multiplication and division methods.

In order clearly to understand this table, we must bear in mind that division is always at least a partial transformation from the denary scale to the scale of notation of which the divisor is the base. The adoption of the denary or decimal scale by all civilized notation is due entirely to the fact that man has ten fingers. There is no other peculiar charm about it; in some respects the duodenary scale would certainly be superior. As a simple example let us divide nine by seven; we get of course once and two over. This means that the magnitude which is represented by 9 in the denary scale is represented by 12 in the septenary scale. In this case the transformation is complete. We may test the accuracy of our work by writing down the successive numbers in the two scales.

Denary	1	2	3	4	5	6	7	8	9
Septenary	1	2	3	4	5	6	10	11	12

Now let us work out the problem on the Soroban. Set down the number 9 with 7 a little to the left. The division table for seven takes no account whatever of the number nine; but it says "shichi shichi shin ga in jū," or as it might be paraphrased, "seven seven gives one ten"—where "ten" signifies not the number but the rod. As the operator repeats this formula, he removes 7 from the nine and pushes 1 up on the next rod to the left. The operation is shown in diagram 1 of Fig. 9.

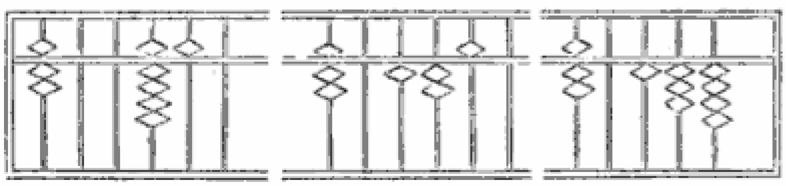


(Fig. 9.)

Now this number, represented by 12 in the septenary scale, we cannot call twelve, because twelve means ten and two, whereas here we have only seven and two. Practically we keep the unit as in the denary scale and use the phrase two-sevenths, which really signifies two in the septenary scale. A more complex example will make it clearer. Let it be required to divide 95 by 7; in other words, how many times is 7 contained in 95. By ordinary processes we obtain 13 and 4 over. This 4 is in the septenary scale; but 13 is still in the denary scale. Hence the transformation is only partial. To complete the transformation into the septenary scale we must express the denary 13 as the septenary 16; so that finally the denary 95=septenary 164. In this septenary number the 6 means 6 sevens, and 1 means 1 seven-sevens; precisely as in the denary number 9 means from its position

9 tens. Practically of course we keep the quotient in the denary scale and say 13 and 4-sevenths. Now perform this on the Soroban. First, as before, we remove 7 from the 9 and move 1 up on the next rod to the left. The Soroban now reads 125 as shown in diagram 2 of Fig. 10.

I. II. III.



(Fig. 10.)

We have now to divide twenty-five by 7. The Soroban manipulator, however, does not look so far ahead, but deals simply with the twenty, or what is the same thing, the 2 on the 'tens' rod. His division table says "Shichi ni ka ka no roku, or as we may paraphrase it, "Seven out of two, add six below", which implies that the 2 is to be left as it is and 6 added to the next rod, to the right. (This is precisely the equivalent of seven out of twenty, twice and six.) Now it is evident at a glance that we cannot add 6 to the next rod, which has already 5 on it. But, bearing in mind that we are still dividing by seven, we remove seven from the overfilled rod and push one up on the "tens' rod. Hence the operator is to add one to the 'tens' rod, remove seven from, and add six to, the 'units' rod; or simply add one to the 'tens' rod and remove one from the 'units' (1=7—6). The general rule is obvious. If the remainder number to be added to any rod equals or exceeds the number of unused counters on that rod, then one counter is pushed up on the rod immediately to the left, and from the first named rod is subtracted that number which with the remainder makes up the divisor. Hence the final result stands as is shown in diagram 3 of Fig. 10, where 4 appears as the remainder.

As another example let us divide 427032 by 8. We may represent the operations symbolically thus, naming the successive results by a, b, c, d, e, f, and drawing a bar to show how far the operation has advanced. The translation of the Japanese verbal accompaniment to these operations is given below:

(8)	4	2	7	0	3	2
a.	5 |	2	7	0	3	2
b.	5	3 |	3	0	3	2
c.	5	3	3 |	6	3	2
d.	5	3	3	7 |	7	2
e.	5	3	3	7	8 |	8
f.	5	3	3	7	9 |	

a. Eight four, replace by 5.

b. Eight two, below add 4 (which being impossible means add 10[14] take off 4).

c. Eight three, below add 6.

d. Eight six, seventy-four.

e. Eight seven, eighty-six.

f. Eight eight, gives one ten.

The chief advantage of the Soroban over ciphering lies in the absence of all mental labour such as is necessarily involved in the "carrying" of the remainder to the next digit. Once the Division Table is mastered and the fingers play obediently to the sound, the whole operation becomes perfectly mechanical. The only disadvantage is the often mentioned one, that the dividend disappears in the process. But this, as we have seen, is a small thing after all.

We shall now go through a problem in long division; and here the process is very similar to our own. Indeed, it can hardly escape notice that short division on the Soroban is essentially the same process as long division with us.

Let it be required to divide 703,314 by 738. Here again we shall symbolically represent the successive operations, so far as is necessary for clearness.

(738)		7	0	3	3	1	4
a.	1 |	0	0	3	3	1	4
b.		9 |	7	3	3	1	4
c.		9 |	3	9	1	1	4
d.		9	5 |	4	1	1	4
e.		9	5 |	2	2	1	4
f.		9	5	2 |	8	1	4
g.		9	5	3 |	1	1	4
h.		9	5	3 |	0	0	0

[14] This 10 is not "ten" but "eight", since for the moment we are working in the octenary scale.

The start is made by consideration of the first figure on the left of the divisor.

a. Seven seven, one ten. Take account now of the next figure in the divisor, multiply it by the 1 already obtained in the quotient and subtract the product from the second place in the dividend. Clearly this is impossible. Now observe that the first two figures of the line opposite *a*, namely 10, are really in the septenary scale.

b. Hence take 1 from 10 (not ten but really seven) and add 7 to the next lower rod.

c. Use 9 as multiplier now; subtract 9 times 30 or 270 from 733 and then 9 times 8 or 72 from the remainder. This completes the first operation, and is essentially the same as the first stage in the ordinary long division method.

d. Start afresh as before with "seven three, forty two."

But 2 is greater than 1, the unused counter on the corresponding rod. Hence add one to 4 on the second rod and subtract 5 (7—2) from the third rod.

e. Use 5 as multiplier; subtract 5 times 30 from 411, and 5 times 8 from the remainder.

f. Start once again with "seven two, add six below."

g. "Seven seven, gives one ten;" which means,—add one to the third rod, subtract seven from the fourth.

h. Use 3 as multiplier; subtract 3 times 30 from 114, and 3 times 8 from the remainder.

Here again in the complete absence of any mental labour lies the peculiar merit of the Soroban. The only operation which calls for special remark is *a*, in which the first figure of the quotient is obtained by a process singularly rapid and free from all concentration of mind.

It is not necessary for rapid manipulation of the Soroban that one who is accustomed to western modes of thought should use the Japanese Division Table. We may substitute our own peculiar method of dividing. There are, however, two of the Japanese Tables which are singularly beautiful in their construction, the one for 5 and the one for 9. For example let us divide 240635 by 5. The Table says "five two, add two," which is exactly the equivalent ultimately of our statement that "five into twenty give four." We may show the process symbolically thus:

(5)	2	4	0	6	3	5
	4 ∣	4	0	6	3	5
	4	8	0 ∣	6	3	5
	4	8	1	2 ∣	3	5
	4	8	1	2	6 ∣	5
	4	8	1	2	7 ∣	

The process simply amounts to multiplying by 2 and dividing by 10; but with the Soroban it is peculiarly rapid.

Again let us divide the same number by 9. The Table says "nine two add two below," which is identical in result with "nines in twenty twice and two," and so with the others. Symbolically we have:

2	4	0	6	3	5
2 ∣	6	0	6	3	5
2	6 ∣	6	6	3	5
2	6	6 ∣			

Here we cannot add 6 below; but instead we take off 3 (9—6) and put on one above as usual. Hence we obtain:—

2	6	7 ∣	3	3	5
2	6	7	3 ∣	6	5
2	6	7	3	7 ∣	2

The 2 is the remainder of course.

Extraction of Square Root (*Kai hei hō*, 開平方).

This requires, as in the ordinary ciphering process, a knowledge of the squares of the nine digits; but its peculiarity lies in the use of another table of half-squares, *Han ku ku* (半九九). In both the Soroban and ciphering processes, the basis is the algebraic truth that the square of a binomial is the sum of the squares of the two components together with twice their product, or the corresponding geometrical theorem that if a straight line be divided into two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts. In the arithmetical extraction of square root, the quantity is considered as consisting of two parts, the first part being that multiple of the highest power of 100 contained in the number which is a complete square. Thus the number 6889 is divided into 6400 and 442. But

$$6400 + 489 = 80^2 + 489$$

so that 80 is the first approximation to the value required. If we compare this with the binominal expression

$$\begin{aligned}(a + b)^2 &= a^2 + 2\,a\,b + b^2 \\ &= a^2 + (2\,a + b)\,b\end{aligned}$$

we see that our next operation must be to form the divisor $2a + b$ that is, in the numerical case $160 +$ a quantity still unknown, but this quantity still unknown is also the quotient of the remainder 489 by the divisor. The process is to use 160 as a trial divisor, so as to get an idea what the unknown quantity may be. In this case we obtain 3, which added to 160 gives 163; and this multiplied by 3 gives 489. Hence the square root of 6889 is 83. Now in this mode of procedure a divisor quite distinct from the final result has to be formed. In the Soroban, however, whose peculiar feature in all operations is the disappearance of the various successive operations as the result is evolved, a distinct divisor does not appear. Thus, by an obvious transformation, we have

$$(a + b)^2 = a^2 + 2\,(a + \tfrac{b}{2})\,b$$

Comparing this as before with

$$6889^2 = 80^2 + 489$$

we see, that by *halving* the remainder 489, we may employ a itself, that is 80, as our trial divisor. In completing this step we must take $\frac{1}{2}\,b^2$ instead of b^2; and hence the importance in the Soroban method of the table of half squares. The simplicity of the method will be recognised from the following example. It is required to extract the square root of 418,609. As in ordinary ciphering, tick off the number in pairs, beginning at the right hand. Then clearly 600 is the first approximation to the value of the square root, or 6 is the first figure in the answer. Move up 6 on a convenient rod somewhat to the left. The successive operations are given symbolically below, the description following as in the previous examples.

	6	4	1	8	6	0	9
a.			5	8	6	0	9
b.			2	9	3	0	4.5
c.	64			5	3	0	4.5
d.				4	5	0	4.5
e.					3	0	4.5
f.						2	4.5
g.	647						0

a. Subtract 6^2 or 36 from 41 leaving 5.

b. Halve the whole remainder 58609.

c. Use 6 as trial divisor of 29. This gives 4. Subtract 4×6 or 24 from 29, leaving 5, and consider 64 as the full divisor.

d. Subtract half the square of 4 from 53. This completes the second stage.

e. Start with 6 again as trial divisor of 45, or more accurately 600 as trial divisor of 4504·5. This gives 7. Subtract 7×6 or 42 from 45.

f. Subtract 7 times 40 from the remainder 304·5.

g. Subtract half the square of 7 from the remainder 24·5. 647 thus appears as the last divisor and, as there is no remainder, it is the square root of 418,609.

The whole process may be easily proved by considering the expansion of the square of a polynomial. Take for example the quadrinomial $(a + b + c + d)$

$$
\begin{aligned}
(a+b+c+d)^2 &= a^2 + b^2 \quad + c^2 \quad + d^2 \\
&\qquad + 2\,ab + 2\,bc + 2\,cd \\
&\qquad\qquad + 2\,ac + 2\,bd \\
&\qquad\qquad\qquad + 2\,ad \\
&= a^2 + 2\,\Big[\left(a + \tfrac{b}{2}\right) b \\
&\qquad + \left(a + b + \tfrac{c}{2}\right) c \\
&\qquad + \left(a + b + c + \tfrac{d}{2}\right) d\Big]
\end{aligned}
$$

Extraction of Cube Root (*Kai ryu hō* 開立方)

The difference in the Soroban and ciphering processes arises from the same cause as in the case of square root. That is, instead of preparing a divisor, the Soroban worker prepares the dividend. The much greater complication in the case of the cube root necessitates an *undoing* of the processes of preparation at each successive stage—a mode of operation which was obviated in the case of square root by the use of the table of half squares. The analogous table of "third cubes" would be excessively awkward in operating with, because of the decimal non-finiteness of the fractions of three. The operator is expected to know by heart the table of cubes, or *Sai jō ku ku* (再乘九九). As in the ordinary ciphering method, the Soroban method depends upon the expression for the cube of a binomial. Consider for example the number 12167. The first operation is to tick off in *threes*, that is in groups of ten-cubed. Now 12 lies between the cubes of 2 and 3. Hence 20 is the first approximation to the cube root of 12167. We have

$$\begin{aligned} 12167 &= 8000 + 4167 \\ &= 20^3 + 4167 \end{aligned}$$

Now comparing this with the expression

$$\begin{aligned} (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + (3a^2 + 3ab + b^2)\,b \end{aligned}$$

we see that we must form a divisor whose most important part is $3a^2$, that is, 3×400 or 1200. Using 1200 as trial divisor of 4167, we get 3, which corresponds to the b in the general expression. We now form the complete divisor by adding to 1200 the expression

$$\begin{aligned} 3ab + b^2 &= 3 \times 20 \times 3 + 3 \times 3 \\ &= 180 + 9 \\ &= 189 \end{aligned}$$

Thus we find as final divisor 1389, which multiplied by 3 gives 4167; and hence 23 is the answer required.

The method on the Soroban depends upon the following transformation of the binomial expression.

$$(a+b)^3 = a^3 + 3a\left(a + b + \frac{b^2}{3a}\right) b$$

Here by dividing the remainder (after subtracting the cube of the first member) by that member and by 3, we obtain an expression whose principal part is ab, that is, the product of the first member and the as yet unknown second member. Hence using a as trial divisor of the first figures of the prepared dividend we get b. In the process, the a or first member of the answer is set down in such a position relatively to the original expression that the b when it *is* finally evolved falls into its proper place succeeding a. We now subtract b^2 from its proper place in the remainder; and the final remainder obtained is $b^3/3a$. Operating upon this by multipliying first by 3 and then by a, that is by an exact reversal of the original process of preparation, we get b^3 left. We shall illustrate the process by extracting the root of 12167 according to the Soroban method. The number is first ticked off by threes in the usual way, and the first member of the answer is set down on the first rod to the left of the highest triplet. In this particular example there are only two significant figures in the highest triplet, so that the 2 is set down two rods to the left of the first figure in the original number. The successive steps are as follows; and as position is of supreme importance in this operation, we shall symbolise the Soroban rods by ruled columns.

a.	2		1	2	1	6	7
b.	2			4	1	6	7
c.	2		2	0	8	3	1
d.	2		6	9	4	3	2
e.	2	3	0	9	4	3	2
f.	2	3			4	3	2
g.	2	3			1	3	1
h.	2	3				2	7
i.	2	3				0	0

a. Tick off into powers of 10^3 and consider the significant figures in the highest triplet, in this case 12. Two rods to the left set down 2, the highest integer whose cube (8) is less than 12.

b. Subtract 2^3 or 8 from 12; or, to be more precise subtract, 20^3 or 8000 from the original number.

c. Divide the remainder by the 2, which is the first found member of the answer. This, in accordance with the Soroban method of division, requires the first figure of the quotient to be set down one rod to the left. Also it must be noted that the last *unit* is a fractional remainder and means really one-half.

d. Divide by 3, carrying out the process until the last rod with the $\frac{1}{2}$ remainder is reached. To this unit the unit of the fraction one-third which appears a final remainder is added; so that the 2 on the last rod really means one-half and one-third. The division by

3 might be stopped at the preceding rod, so that instead of 69432 we should have 69411, in which the first unit means $\frac{1}{3}$ and the second $\frac{1}{2}$. There is greater chance of confusion, however, in this method than in the one shown, as will be seen when we come to the later stages.

e. Divide by 2, but stop when the first figure in the quotient, in this case 3, is obtained.

f. Continue this operation of division, regarding the newly obtained 3 as part of the divisor; or in other words, subtract 3^2 or 9 from the next place to the right. We have now left a remainder represented by 43 and $\frac{1}{3}$ and $\frac{1}{2}$. This remainder is of the form $\frac{b^3}{a^3}$; and to bring it back to a workable form we must multiply it by $3a$. We must be careful, however, to do this so as to take proper account of the peculiar mixed fraction represented by 2 on the last rod to the right. The next two stages effect this.

g. Multiply by 3, beginning, however, at the second last rod, and thus undoing the operation *d*. Multiplication on the Soroban is accompanied by displacement to the right. Hence the product 3×43 or 129 has its last right-hand figure added to the rod containing the mixed remainder 2; and the final result of this operation gives 131, in which the last unit means as before one-half.

h. Multiply by 2, beginning with the second last rod, and thus undoing the effect of operation *c*. The product 2×13 or 26 is added to the 1, and the 27 appears as the final expression.

i. Subtract 3^3 or 27, and the remainder is zero.

Had we stopped in the operation *d* at an earlier point as suggested, we should have had to modify the reverse operation *g*. Thus, only the 4 of 411 would need to be multiplied by 3, giving of course 12 to be added to the first of the two units. The final result would have been of course 131, as already obtained.

As a further illustration of the method, let us take the case of a much larger number. It is required to find the cube root of 237,176,659. We shall divide the operation into two stages, the first of which corresponds with the simpler example already given.

a.	6	2	3	7	1	7	6	6	5	9
b.	6		2	1	1	7	6	6	5	9
c.	6		3	5	2	9	2	6	5	9
d.	6	1	1	7	6	3	3	6	5	9
e.	6	1	5	6	6	3	3	6	5	9
f.	6	1	1	6	9	9	2	6	5	9
g.	6	1	1	0	1	9	6	6	5	9
h.	6	1	1	0	1	9	5	6	5	9

a. Tick off the number in triplets beginning at the "units" place, and find the nearest complete cube to 237. It is clearly 216, the cube of 6. Set down 6 immediately to the left of 237.

b. Subtract 216 from 237.

c. Divide the remainder to the end of the next triplet by 6. It is unnecessary to go further in the division. Such an extending of the process would simply give unnecessary extra work in the reverse operations. The 2 in the last place of the second triplet is as before a fractional remainder and means *two-sixths*.

d. Divide by 3, manipulating the remainder as in the previous example.

e. Divide by 6 as trial divisor, giving 1 for quotient, and continue the division with 61. In other words subtract 1×61 from 117.

f. Undo the effect of *d* by multiplying 5663 by 3 and adding in the mixed remainder 3.

g. Undo the effect of *c* by multiplying 1699 by 6 and adding in the remainder 2.

h. Subtract 1^3 or 1 from 10196.

The final result then is 61 and a remainder 10195659, which must now be treated so as to obtain the third figure in the required answer. This second stage is exactly similar to the first stage after operation *b*. The steps are as follows:—

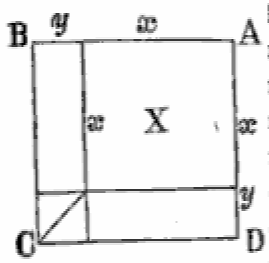
h.	6	1	1	0	1	9	5	6	5	9
i.	6	1	1	6	7	1	4	1	5	8
j.	6	1	5	5	7	1	3	6	7	8
k.	6	1	9				3	6	7	8
l.	6	1	9				1	1	5	8
m.	6	1	9					7	2	9
n.	6	1	9					0	0	0

i. Divide the remainder 10195659 by 61. The two last figures, 58, form a fractional remainder and mean 58/61.

practised on the Chinese abacus, we cannot but be struck with their singular beauty and compactness. Once the meaning of the indications is understood, the operations of addition and subtraction are self-evident. Multiplication and division are of course in the first place repetitions of addition and subtraction. Thus if we wish to know how many times six is contained in 40, we have simply to go on subtracting successive sixes till no amount of the value of a six remains. We find we have to do this 6 times in succession and have 4 left after all; hence we say 6 out of 40, 6 times and 4 over. If we have this operation to perform frequently, it is of distinct time-saving advantage to stow it away in our memory. It is in this way that multiplication and division table have been found a practical necessity.

It has been already pointed out that the division table is a peculiar feature in the manipulation of the abacus as used in China and Japan. We have nothing corresponding to it in our western methods. With us the art of division is developed from a previous knowledge of the multiplication table. The mental process by which a beginner discovers how many times 38 contains 7 is to run up the multiplication table till a multiple is reached which is less than 38 by a number less than seven. Thus he finds 35, which is 5 times 7, and which differs from 38 by 3. With practice the finding of the necessary multiple becomes almost instantaneous; and the average school-boy is hardly conscious of the successive mental operations of multiplication and subtraction by which he effects division. With the Soroban worker, however, it is quite otherwise. He learns a division table, of quite a conventional construction. In reality he learns the result of dividing the pure decade numbers by the simple digits; but instead of saying "seven into forty, five and five," he says "seven four, fifty-five." Such a convention, strange though it may sound, is peculiarly suitable for Abacus use. Upon it indeed may be said to depend largely the wonderful efficiency of the instrument. Exactly by what process of development the division table in its perfected form was evolved, is a problem which will probably never receive a solution; but it is clearly of purely Abacus origin.

The processes for extracting square root and cube root, on the other hand, imply a knowledge of mathematics much wider than the abacus itself could ever teach. Square Root might perhaps have been evolved as a purely arithmetical operation on the abacus; but Cube Root certainly could not. It seems more reasonable to suppose that both processes were deduced by some more general mathematical method, either algebraic or geometric. The geometrical aspect is indeed most instructive. Consider for example the square A B C D, from which has been subtracted the small square X, whose side x is known in finite terms. The L-shaped portion measures the remainder after X has been subtracted from the large square. From this remainder we have to find the length y, which with x makes up the side of the large square. The line drawn from C to the contiguous corner of X evidently cuts the L-shaped remainder into two halves. And each half is made up of the product of x and y and half the square of y. Here we have at once the suggestion of the abacus rule for extracting square root. A similar consideration of the properties of the cube would lead to the abacus rule for extracting the cube root. It is not probable however that these rules were discovered in this way. They are rather to be regarded as having been deduced from general algebraic considerations, just as our own rules are. They involve a knowledge of the binomial theorem, not necessarily in its complete generality, but so far at least as positive integers are concerned. It is known, however, that Chinese mathematicians have been acquainted for centuries with the binomial theorem, which they employed in the solution of equation of high degree. Hence it is almost certain that the abacus rule for cube root is a formula deduced from the algebraic mode of solving such an equation as



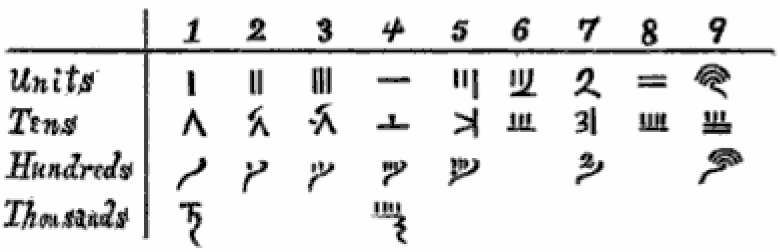
$$x^3 - a = 0$$

The rule of course had to be formulated so as to suit the peculiar conditions of the arithmetic abacus. The discussion of what might be called the algebraic abacus or chess-board like arrangement for solving equations, is beyond the scope of the present paper.

PLATE I.

Symbolic Stage

The Hieratic Numerals

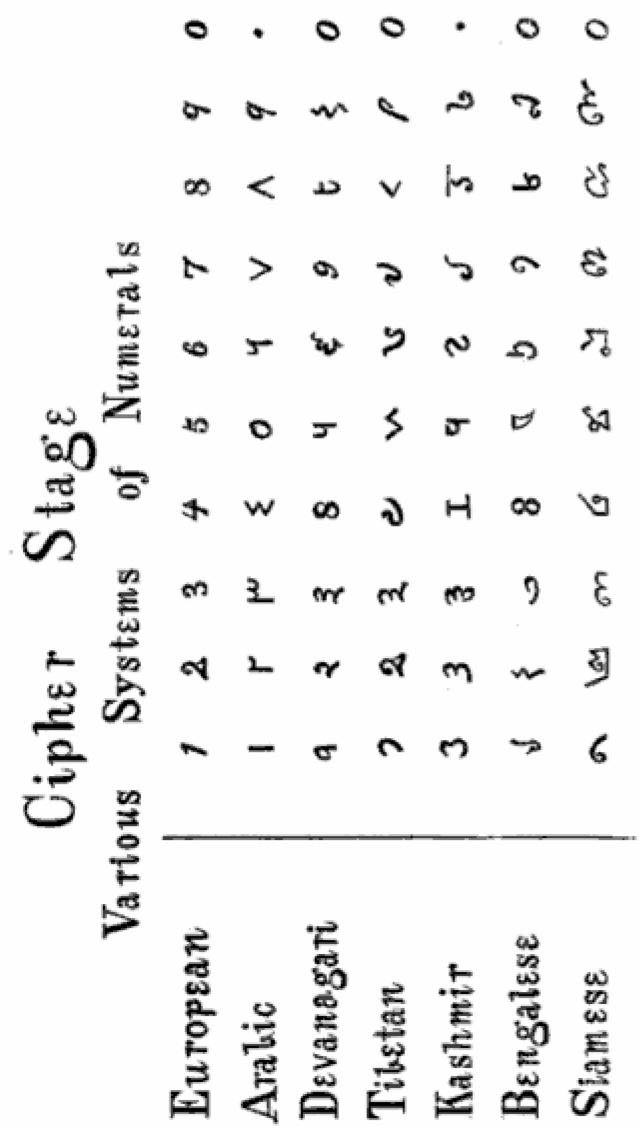


Decimal Stage

The Chinese and Tamil Numerals

	C	T		C	T
1	一	௧	*10*	十	௰
2	二	௨	*100*	百	௱
3	三	௩	*1000*	千	௲
4	四	௪			
5	五	௫	*26*	二十六	௨௰௬
6	六	௬			
7	七	௭			
8	八	௮	*734*	七百三十四	௭௱௩௰௪
9	九	௯			

PLATE II.


Cipher Stage
Various Systems of Numerals
European
Arabic
Devanagari
Tibetan
Kashmir
Bengalese
Siamese
1 2 3 4 5 6 7 8 9 0