THE CALCULATING MACHINE OF THE EAST: THE ABACUS

(Abridged from the article "The Abacus in its Historic and Scientific Aspects" in the *Transactions of the Asiatic Society of Japan*, vol. xiv., 1886)

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The Abacus possesses besides a high respectability, arising from its great age, its widespread distribution, and its peculiar influence in the evolution of our modern system of arithmetic. In the Western lands of to-day it is used only in infant schools, and is intended to initiate the infant mind into the first mysteries of numbers. The child, if he ever is taught by its means, soon passes from this bead-counting to the slate and slate pencil. He learns our Indian Numerals, of which *one* only is at all suggestive of its meaning; and with these symbols he ever after makes all his calculations. In India and all over civilised Asia, however, the Abacus still holds its own; and in China and Japan the method of using it is peculiarly scientific. It seems pretty certain that its original home was India, whence it spread westward to Europe and eastward to China, assuming various forms, no doubt, but still remaining essentially the same instrument. Its decay in Europe can be traced to the gradual introduction and perfecting of the modern cipher system of notation, which again in part owes its early origin to the indications of the Abacus itself.

The *Soroban* or Japanese Abacus is one of the first objects that strongly attracts the attention of the foreigner in Japan. He buys at some shop a few trifling articles and sums up the total cost in his own mind. But the tradesman deigns not to perplex himself by a process of mental arithmetic, however simple. He seizes his Soroban, prepares it by a tilt and a rattling sweep of his hand, makes a few rapid, clicking adjustments, and names the price. There seems to be a tradition amongst foreigners that the Soroban is called into requisition more especially at times when the tradesman is meditating imposition; and in many cases it is certain that the Western mind, with its power of mental addition, regards the manipulator with a slight contempt. A little experience, however, should tend to transform this contempt into admiration. For it may be safely asserted that even in the simplest of all arithmetical operations the Soroban possesses distinct advantages over the mental or figuring process. In a competition in simple addition between a "Lightning Calculator," an accurate and rapid accountant, and an ordinary Japanese small tradesman, the Japanese with his Soroban would easily carry off the palm.

Summary of Part I.: The Historic Aspect

The Abacus, as used in China and Japan, bears, on the very face of it, evidence of a foreign origin. The numbers are set down on it with the larger denomination to the left, a result which could come from a people either speaking and writing inversely, or speaking and writing directly. Historically, the home of the Abacus is in India; but it could hardly have been invented by the Aryan Indians, who wrote directly and spoke inversely. The probability is they borrowed it from Semitic peoples, who were the traders of the ancient world; and these may have invented it, or, as is perhaps more probable, received it from a direct-speaking, direct-writing race, such as we know the highly cultured Accadians to have been.

In early times the Abacus, as being an evolution from the natural Abacus—the human hand—pursued a course of development entirely different from that of the graphic representation of numbers. This latter we can trace through four stages—the Pictorial, the Symbolic, the Decimal, and the Cipher. The Pictorial we find in the Egyptian hieroglyphics, the Accadian Cuneiform, and the technical Chinese of mathematical treatises; the Symbolic in the numerous methods which grew up with the development of alphabets and syllabaries; and the Decimal in the simplifications of these, which live to-day in the Chinese and Tamilic systems. Once the Decimal stage was reached, its general similarity to the Abacus indications suggested bringing them into still closer correspondence.

This advance seems to have taken place amongst the Aryan Indians, who, along with the Aryans of the West, very soon discarded the Abacus for the more convenient Cipher notation. With the Chinese, Tamils and Malayalams of South India, no advance was made in this direction; the reason being simply that the Abacus better suited their numeration. These peoples speak directly, so that their nomenclature fits in perfectly with the Abacus indications, and makes its manipulation more rapid and certain than calculation by ciphering. An Aryan Indian with his inverse speaking could never work the Abacus with the same facility as a Japanese unless he worked from right to left—a mode of procedure quite foreign to his nature. It is not so foreign to Chinese and Japanese, however, to work from left to right, as each individual character is formed in this way. It may be safely concluded that only amongst a people who used the direct mode of naming numbers, or who with the inverse mode of naming preferred the inverse mode of manipulating, could the Abacus in the form in which it was evolved ever attain the beauty of action of the Japanese Soroban. To the discussion of its peculiar merits we now proceed. We shall employ throughout the Japanese name, which it should be noted is simply a mispronunciation of the Chinese name—*Swanpan*.

PART II.: THE SCIENTIFIC ASPECT

The Soroban may be defined as an arrangement of movable beads, which slip along fixed rods and indicate by their configuration some definite numerical quantity. Its most familiar form is as follows. A shallow rectangular box or framework is divided longitudinally by a narrow ridge into two compartments, of which one is roughly some three or four times larger than the other. Cylindrical rods placed at equal intervals apart pass through the ridge near its upper edge, and are fixed firmly into the bounding sides of the framework. On these rods the counters are "beaded." The size of the counters determines the interval between the rods, the number of which will of course vary with the length of the framework. Each counter (Japanese *tama*, or ball) is radially symmetrical with respect to its rod, on which it slides easily. Looked at from in front of the box, the form in perspective is that of a rhombus, the rod passing through the blunt angles. This double cone form makes manipulation rapid, the finger easily catching the ridge-like girth of the *tama*.

On each rod there are six (sometimes seven) *tama*. Five of these slide on the longer segment of the rod, the remaining one (or two) on the shorter. When the *tama* on any

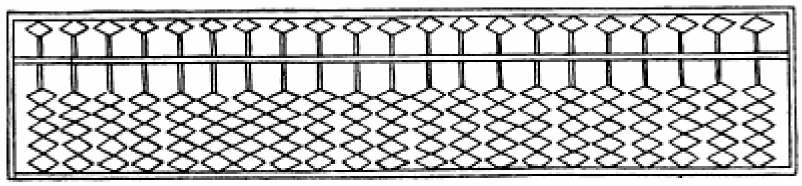

FIG. 1.

segment of a rod are set in close contact, a part of the rod is left bare. The length of this bare portion is determined by a double consideration. It must be long enough to be clearly visible, and yet not so long as to make the action of the fingers irksome by reason of excessive stretching.

When a Soroban is lifted indiscriminately, the counters will take some irregular configuration upon their rods, being limited in their motions by the bounding walls and the dividing ridge. To prepare it for use, the framework is tilted slightly with the smaller compartment uppermost, so that each set of five counters slips down to the bounding wall end of its rod and each single counter[1] on its short rod slips down upon the upper surface of the dividing ridge. The framework is then gently adjusted till all the rods become horizontal, so that if any counter is shifted it will have no tendency to move back to its former position. By a sweep of the finger-tips along the surfaces of the single counters, these are driven from their contact with the dividing ridge to the other extremities of the rods. In this configuration, in which the counters are all as far away as possible from the dividing ridge, the Soroban is prepared for action. The number represented is zero. This position is shown in fig. 1.

Let now any first counter of a set of five be moved till it is stopped by the ridge, as shown in the first diagram of fig. 2. This will represent 1, 10, 100, 1000, etc., as may be desired. Let it represent 1, then a second moved up will give us 2, a third 3, a fourth 4. This last is shown in the second diagram of fig. 2. The last moved up will of course give 5; but this number is also given by pushing back the five counters to their zero position and bringing down the corresponding single counter to the ridge. This is shown in the last diagram of fig. 2.

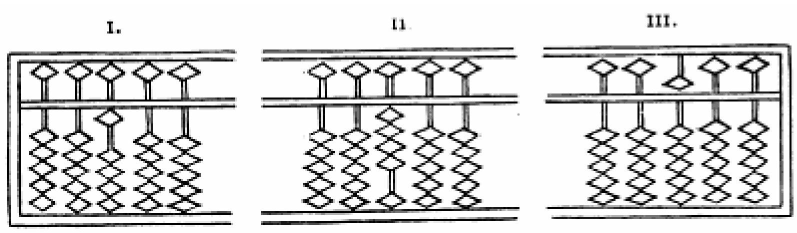

FIG. 2.

[1] We shall henceforth only speak of *one* counter as being on the short rod. The two counters, although facilitating somewhat certain operations in division, are not really necessary, and their use is exceptional.

Leaving this single one in position, we get 6 by pushing up 1, 7 by pushing up 2, and so on till 9 is reached, as shown in fig. 3. The number 10 is then represented either by moving up the last counter, or more usually by clearing the rod of all its counters and moving one up on the next rod to the left, as shown also in fig. 3.

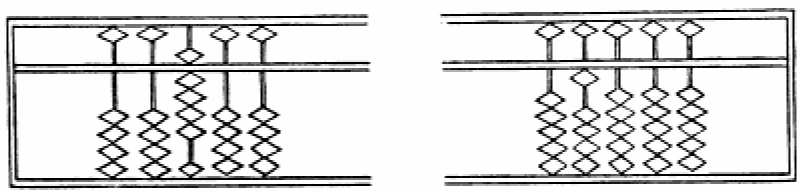

FIG. 3.

The mode of representing any number is thus obvious, being simply a mechanical model of our cipher system. Each rod corresponds to a definite figure "place" (Japanese *Kurai*) or power of ten. One being first chosen as the unit, the next to the left is the "tens," the next the "hundreds," the next the "thousands," and so on; while the successive rods to the right will represent the successive decimal places—tenths, hundredths, thousandths, etc. When the counters are as far as possible from the dividing ridge they have no value; when they are pushed as *near* the ridge as possible they have values as already indicated. The single counter when pushed down upon the ridge has five times the value of any other counter upon that rod. In fig. 4 the number 3085.274 is shown. The mark V is placed over the "units" rod.

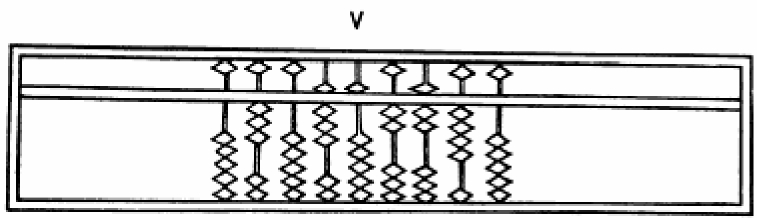

FIG. 4.

The operations of addition and subtraction are self-evident. Thus, let it be required to add to this number 352.069. On the "hundreds" rod push up 3; and proceed throughout whenever it can be done this way. On the "tens" rod, however, where only two counters are left, it is impossible to push up 5. But since 50=100-50, the addition is effected by pushing up one counter on the "hundreds" and removing 5 from the "tens" rod. This gives of course 4 on the "hundreds" rod and leaves 3 on the "tens." Then push up 2 on the "units" rod; then 1 on the "tenths" rod with a simultaneous removal of 4 from the "hundredths" rod, since 10-6=4; then 1 on the "hundredths" rod with a simultaneous removal of 1 from the "thousandths" rod. The final result 3437.343 is given in fig. 5.

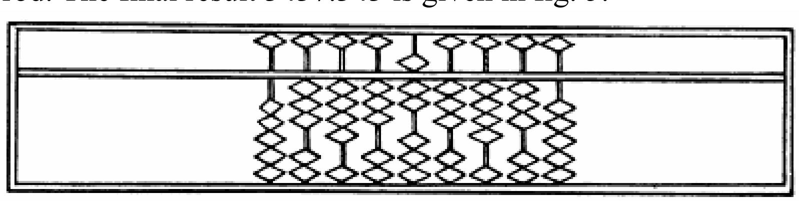

FIG. 5.

Subtraction is executed in a similar manner. It will be noticed that these operations involve no mental labour beyond that of remembering the complementary number, that is, the number which with the given number makes up 10. A glance at the configuration on any rod is sufficient to show if the addition (or subtraction) of a named number can be effected on it; and if this cannot be, it is necessary simply to add (or subtract) one to (or from) the next higher place and subtract (or add) the complementary number from (or to) the place in question. In first experimenting with the Soroban, an operator who is accustomed only to our Western modes of figuring is apt to add mentally, and then set down the result on the instrument. Such a mode is inferior of course to the ordinary figuring method, being liable to error, inasmuch as the number that is being added is not visible to the eye at any time, and the number that it is being added to disappears in the operation. But if anyone will take the trouble to dispossess himself of his Western methods and work in the manner indicated, he will find Soroban addition and subtraction both more rapid and more certain, because attended by less mental exertion, than in figuring. The one seeming disadvantage in the Soroban is that the final result of each step alone appears, so that if any error is made, the whole operation must be carried through from the beginning again. Almost all writers on China or Japan, who have noticed the instrument, bring this forward as a serious disadvantage. But such a conclusion is a hasty one, and shows the writer to possess but small acquaintance with Soroban methods, and little regard to the true aim of calculation. For after all it is the result we wish; and if an error has been made, repetition is necessary both with Soroban and ciphering. The mean position of an accidental error is of course half-way through; and this would tell in favour of the ciphering system. But, on the other hand, the Soroban is, on the average, much more rapid than ciphering, and less liable to error. Only a lengthened series of comparative experiments could establish whether there is any real disadvantage at all.

MULTIPLICATION

Multiplication on the Soroban differs but slightly from our own methods, being effected by means of a Multiplication Table—*ku ku gō sū*,[2] literally, nine-nine combining number. Two peculiarities distinguish this table from ours. First, there is a complete lack of interpolated words like our "times," the multiplier, multiplicand, and product being mentioned in unbroken succession; and second, the multiplier, that is the first-named number, is always the smaller. Thus the multiplication table for six runs:

ichi	roku	roku
ni	roku	jū ni
san	roku	jū hachi
shi	roku	ni jū shi
go	roku	san jū
roku	roku	san jū roku
roku	shichi	shi jū ni
roku	hachi	shi jū hachi
roku	ku	go jū shi

2 Generally called simply ku ku.

It is unnecessary to go to 12 as we do. Knowledge of a multiplication table for any number higher than 9 would retard Soroban manipulation. We British at least are often compelled to learn up to 12 because of our monetary system; and it is often serviceable to know the table for 16. One is early struck by the inability of most Japanese students to multiply by 12 or even 11 in one line.

In multiplying two numbers together on the Soroban, the operator sets the two numbers somewhat apart on the instrument, the multiplier being to the left, the multiplicand to the right. There must be left to the right of the multiplicand a sufficient number of empty rods, a number at least equal to the number of places in the multiplier. The operation is essentially the same as ours; only, instead of multiplying the multiplicand by each figure of the multiplier as we do, the Japanese multiplies the multiplier by each figure of the multiplicand. As the operation goes on the multiplicand gradually disappears, so that finally only the multiplier and product are left on the board. An example will render the method clear. Let it be required to multiply 4173 by 928. Set these on the Soroban, the multiplier anywhere to the left, and 3 empty rods at least to the right of the multiplicand. Henceforward in the diagrams we shall represent visually only the counters which happen to be in use.

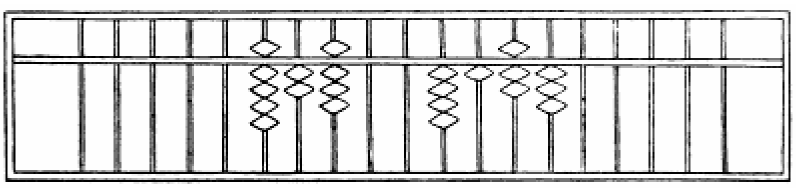

FIG. 6.

Multiply 8 by 3 and set 24 on the Soroban so that the 4 lies just as many places to the right of the multiplicand 3 as there are figures in the multiplier. This 4 is of course in the “units” place of the product; and we shall continue to name the other places accordingly. Next multiply the 2 by 3, and add the product 6 to the “tens” rod. This gives us the result so far 84. Lastly, multiply 9 by 3. This requires 7 to be added to the “hundreds’ rod, and 2 to the “thousands” rod. But before this latter operation can be done, the “thousands” rod must be cleared of its multiplicand 3, which having completely served its purpose may easily be removed, and indeed is better away. Since 3 is to be removed and 2 added, it is sufficient to remove 1 and leave 2. The result so far is shown in fig. 7.

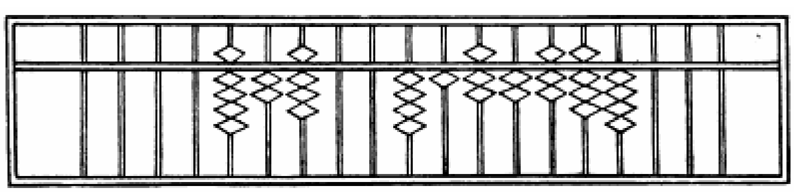

FIG. 7.

Now proceed to multiply with the next figure of the multiplicand, 7, namely:—7 x 8 = 56, of which 5 is to be added to the “hundreds,” and 6 to the “tens” rod; 7 x 2 = 14, that is, 1 to the “thousands,” 4 to the “hundreds”; 7 x 9 = 63, that is, leave 6 on the “ten thousands” rod by taking off 1 from the 7 and add 3 to the thousands. The result of this operation is given in fig. 8

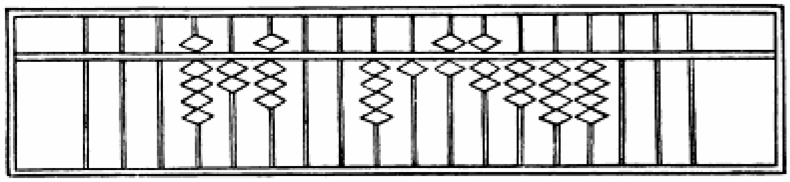

FIG. 8.

The operations with 1 and 4 are similarly carried out, care being taken to add the numbers which make up each several product in their proper places, and to suppress the multiplicand figure at the final operation with the same. The final result is given in fig. 9.

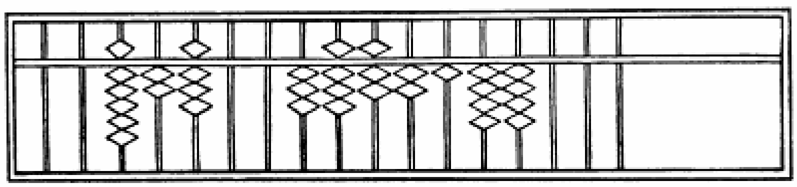

FIG. 9.

It will be noticed that in all addition and subtraction processes the number is added to or taken from the rod rather than from the number on the rod. The eye can tell at a glance if this operation can be effected on the rod in question, or if the next rod to the left has to be called into play. Mental labour is thus reduced to a minimum. The operator hears or utters a certain sound, which means one of two operations. A glance shows which of these it must be; and the fingers execute a certain mechanical movement which accompanies the sound of the words as naturally as the fingers of a pianist obey the graphic commands of a Sonata.

We see then how well fitted for Soroban use is the Chinese and Japanese nomenclature of the numerals; and how ill adapted all such systems must be which say sixteen and five-and-twenty or even sixteen and twenty-five instead of “teen-six” and twenty-five.

DIVISION

Division on the Soroban, although essentially the same as our own Long Division, is in many aspects peculiar and almost fascinating. The *art* of it is based upon a Division Table, called the *ku ki hō*, or Nine Returning Method, which is learned off by heart. This we give in full, with an accompanying translation as literal as possible.

Division Table for Ichi *(one).*

ichi is shin ga in jū	one one gives one ten
“ ni “ “ ni “	one two “ two tens
“ san “ “ san “	“ three “ three “
and so on to	
ichi ku shin ga ku jū	one nine gives nine <tens>

Division Table for Ni *(two).*

ni ichi ten saku no go	two one replace by five
“ ni shin ga in jū	“ two gives one ten
“ shi “ “ ni jū	“ four “ two tens
“ roku “ “ san jū	“ six “ three “
“ has “ “ shi jū	“ eight “ four “

This table could well stop at “ni ni shin ga in jū,” since the higher ones are simply combinations of the first two. This is recognised by the absence of the “two five” statement.

Division Table for San *(three).*

san ichi san jū no ichi	three one thirty-one
“ ni roku “ “ ni	“ two sixty-two
“ san shin ga in jū	“ three gives one ten

The rest is obvious, being indeed but a repetition of the first three statements.

Division Table for Shi *(four).*

shi ichi ni jū no ni	four one twenty-two
“ ni ten saku no go	“ two replace by five
“ san shichi jū no ni	“ three seventy-two
“ shi shin ga in jū	“ four gives one ten

Division Table for Go *(five).*

go ichi ka no ichi	five one add one
“ ni “ “ ni	“ two “ two
“ san “ “ san	“ three “ three
“ shi “ “ shi	“ four “ four
“ go shin ga in jū	“ five gives one ten

Division Table for Roku *(six).*

roku ichi ka ka no shi	six one below add four
“ ni san jū no ni	“ two thirty-two
“ san ten saku no go	“ three replace by five
“ shi roku jū no ni	“ four sixty-four
“ go hachi jū no ni	“ five eighty-two
“ roku shin ga in jū	“ six gives one ten

Division Table for Shichi *(seven).*

shichi ichi ka ka no san	seven one below add three
“ ni “ “ “ roku	“ two “ “ six
“ san shi jū no ni	“ three forty-two
“ shi go jū no go	“ four fifty-five
“ go shichi jū no ichi	“ five seventy-one
“ roku hachi jū no shi	“ six eighty-four
“ shichi shin ga in jū	“ seven gives one ten

Division Table for Hachi *(eight).*

hachi ichi ka ka no ni	eight one below add two
“ ni “ “ “ shi	“ two “ “ four
“ san “ “ “ roku	“ three “ “ six
“ shi ten saku no go	“ four replace by five
“ go roku jū no ni	“ five sixty-two
“ roku shichi jū no shi	“ six seventy-four
“ shichi hachi jū no roku	“ seven eighty-six
“ hachi shin ga in jū	“ eight gives one ten

Division Table for Ku *(nine).*

ku ichi ka ka no ichi	nine one below add one
“ ni “ “ “ ni	“ two “ “ two
“ san “ “ “ san	“ three “ “ three
and so on to	
ku hachi ka ka no hachi	nine eight below add eight
“ ku shin ga in jū	“ nine gives one ten

[In practice some of these phrases are contracted, such as *nitchin in jū* instead of *ni ni shin ga in jū, roku chin in jū* for *roku roku shin ga in jū*, and the like. The two words *ka ka* are run into one, *kakka*, the double k being strongly pronounced as in Italian. (Added, 1914.—C. G. K.)]

It will be noticed that the essential parts of the division tables take no account of the division of a number higher than the divisor. Hence in division, the larger number is named first; whereas in multiplication, as we saw above, the small number is named first. Thus the Japanese gets rid of such interpolated words as “times” and “into” or “out of,” which are necessary parts of our multiplication and division methods.

In order clearly to understand this table, we must bear in mind that division is always at least a partial transformation from the denary scale to the scale of notation of which the divisor is the base. The adoption of the denary or decimal scale by all civilized notation is due entirely to the fact that man has ten fingers. There is no other peculiar charm about it; in some respects the duodenary scale would certainly be superior. As a simple example let us divide nine by seven; we get of course once and two over. This means that the magnitude which is represented by 9 in the denary scale is represented by 12 in the septenary scale. In this case the transformation is complete. We may test the accuracy of our work by writing down the successive numbers in the two scales.

Denary	1	2	3	4	5	6	7	8	9
Septenary	1	2	3	4	5	6	10	11	12

Now let us work out the problem on the Soroban. Set down the number 9 with 7 a little to the left. The division table for seven takes no account whatever of the number nine; but it says “shichi shichi shin ga in jū,” or, as it might be paraphrased, “seven seven gives one ten”—where “ten” signifies not the number but the rod. As the operator repeats this formula, he removes 7 from the nine and pushes 1 up on the next rod to the left. The operation is shown in diagram 1 of fig. 10.

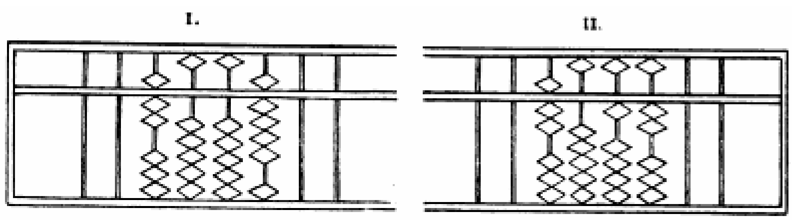

FIG. 10.

Now this number, represented by 12 in the septenary scale, we cannot call twelve, because twelve means ten and two, whereas here we have only seven and two. Practically we keep the unit as in the denary scale and use the phrase two-sevenths, which really signifies two in the septenary scale. A more complex example will make it clearer. Let it be required to divide 95 by 7; in other words, how many times is 7 contained in 95. By ordinary processes we obtain 13 and 4 over. This 4 is in the septenary scale; but 13 is still in the denary scale. Hence the transformation is only partial. To complete the transformation into the septenary scale we must express the denary 13 as the septenary 16; so that finally the denary 95=septenary 164. In this septenary number the 6 means 6 sevens, and 1 means 1 seven-sevens; precisely as in the denary number 9 means from its position 9 tens. Practically, of course, we keep the quotient in the denary scale and say 13 and 4-sevenths. Now perform this on the Soroban. First as before, we remove 7 from the 9 and move 1 up on the next rod to the left. The Soroban now reads 125, as shown in diagram 2 of fig. 11.

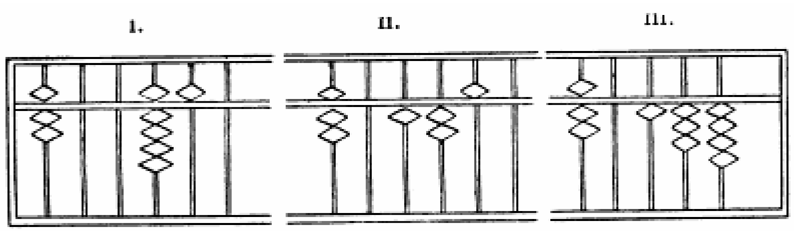

FIG. 11.

We have now to divide 25 by 7. The Soroban manipulator, however, does not look so far ahead, but deals simply with the 20, or, what is the same thing, the 2 on the "tens" rod. His division table says "Shichi ni ka ka no roku," or, as we may paraphrase it, "Seven out of two, add six below," which implies that the 2 is to be left as it is and 6 added to the next rod, to the right. (This is precisely the equivalent of seven out of twenty, twice and six.) Now it is evident at a glance that we cannot add 6 to the next rod, which has already 5 on it. But, bearing in mind that we are still dividing by seven, we remove seven from the overfilled rod and push one up on the "tens" rod. Hence the operator is to add one to the "tens" rod, remove seven from, and add six to the "units" rod; or simply add one to the "tens" rod and remove one from the "units" (1=7 - 6). The general rule is obvious. If the remainder number to be added to any rod equals or exceeds the number of unused counters on that rod, then one counter is pushed up on the rod immediately to the left, and from the first-named rod is subtracted that number which with the remainder makes up the divisor. Hence the final result stands as is shown in diagram 3 of fig. 11, where 4 appears as the remainder.

As another example let us divide 427,032 by 8. We may represent the operations symbolically thus, naming the successive results by *a, b, c, d, e, f*, and drawing a bar to show how far the operation has advanced. The translation of the Japanese verbal accompaniment to these operations is given below:

(8)	4	2	7	0	3	2
(a)	5	2	7	0	3	2
(b)	5	3	3	0	3	2
(c)	5	3	3	6	3	2
(d)	5	3	3	7	7	2
(e)	5	3	3	7	8	8
(f)	5	3	3	7	9	

(a) Eight four, replace by 5.
(b) Eight two, below add 4 (which being impossible means add 10[3] take off 4).
(c) Eight three, below add 6.
(d) Eight six, seventy-four.
(e) Eight seven, eighty-six.
(f) Eight eight, gives one ten.

The chief advantage of the Soroban over ciphering lies in the absence of all mental labour such as is necessarily involved in the "carrying" of the remainder to the next digit. Once the Division Table is mastered and the fingers play obediently to the sound, the whole operation becomes perfectly mechanical. The only disadvantage is the often mentioned one, that the dividend disappears in the process. But this, as we have seen, is a small thing after all.

We shall now go through a problem in long division; and here the process is very similar to our own. Indeed, it can hardly escape notice that short division on the Soroban is essentially the same process as long division with us.

Let it be required to divide 703,314 by 738. Here again we shall symbolically represent the successive operations, so far as is necessary for clearness.

(738)		7	0	3	3	1	4
(a)	1	0	0	3	3	1	4
(b)		9	7	3	3	1	4
(c)		9	3	9	1	1	4
(d)		9	5	4	1	1	4
(e)		9	5	2	2	1	4
(f)		9	5	2	8	1	4
(g)		9	5	3	1	1	4
(h)		9	5	3	0	0	0

The start is made by consideration of the first figure on the left of the divisor.

(a) Seven seven, one ten. Take account now of the next figure in the divisor, multiply it by the 1 already obtained in the quotient and subtract the product from the second place in the dividend. Clearly this is impossible. Now

3 This 10 is not "ten" but "eight" since for the moment we are working in the octenary scale.

observe that the first two figures of the line opposite *a*, namely 10, are really in the septenary scale.

(b) Hence take 1 from 10 (not ten but really seven) and add 7 to the next lower rod.

(c) Use 9 as multiplier now; subtract 9 times 30 or 270 from 733 and then 9 times 8 or 72 from the remainder. This completes the first operation, and is essentially the same as the first stage in the ordinary long division method.

(d) Start afresh as before with "seven three, forty two."

> But 2 is greater that 1, the unused counter on the corresponding rod. Hence add one to 4 on the second rod and subtract 5 (7 – 2) from the third rod.

(e) Use 5 as multiplier; subtract 5 times 30 from 411, and 5 times 8 from the remainder.

(f) Start once again with "seven two, add six below."

(g) "Seven seven, gives one ten," which means—add one to the third rod, subtract seven from the fourth.

(h) Use 3 as multiplier; subtract 3 time 30 from 114, and 3 times 8 from the remainder.

Here again in the complete absence of any mental labour lies the peculiar merit of the Soroban. The only operation which calls for special remark is *a*, in which the first figure of the quotient is obtained by a process singularly rapid and free from all concentration of mind.

It is not necessary for rapid manipulation of the Soroban that one who is accustomed to Western modes of thought should use the Japanese Division Table. We may substitute our own peculiar method of dividing. There are, however, two of the Japanese tables which are singularly beautiful in their construction, the one for 5 and the one for 9. For example, let us divide 240,635 by 5. The table says "five two, add two," which is exactly the equivalent ultimately of our statement that "five into twenty give four." We may show the process symbolically thus:—

(5)	2	4	0	6	3	5
	4	4	0	6	3	5
	4	8	0	6	3	5
	4	8	1	2	3	5
	4	8	1	2	6	5
	4	8	1	2	7	

The process simply amounts to multiplying by 2 and dividing by 10; but with the Soroban it is peculiarly rapid.

Again let us divide the same number by 9. The table says "nine two add two below," which is identical in result with "nines in twenty twice and two," and so with the others. Symbolically we have:—

2 4 0 6 3 5

$$\begin{array}{cccccc} 2\,| & 6 & 0 & 6 & 3 & 5 \\ 2 & 6\,| & 6 & 6 & 3 & 5 \\ 2 & 6 & 6\,| & & & \end{array}$$

Here we cannot add 6 below; but instead we take off 3 (9 – 6) and put on one above as usual. Hence we obtain:—

$$\begin{array}{cccccc} 2 & 6 & 7\,| & 3 & 3 & 5 \\ 2 & 6 & 7 & 3\,| & 6 & 5 \\ 2 & 6 & 7 & 3 & 7\,| & 2 \end{array}$$

The 2 is the remainder of course.

EXTRACTION OF SQUARE ROOT (*Kai hei hō*)

This requires, as in the ordinary ciphering process, a knowledge of the squares of the nine digits; but its peculiarity lies in the use of another table of half-squares, *Han ku ku*. In both the Soroban and ciphering processes, the basis is the algebraic truth that the square of a binomial is the sum of the squares of the two components together with twice their product, or the corresponding geometrical theorem that if a straight line be divided into two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts. In the arithmetical extraction of square root, the quantity is considered as consisting of two parts, the first part being that multiple of the highest power of 100 contained in the number which is a complete square. Thus the number 6889 is divided into 6400 and 489. But

$$6400 + 489 = 80^2 + 489$$

so that 80 is the first approximation to the value required. If we compare this with the binomial expression

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + (2a + b)b\end{aligned}$$

we see that our next operation must be to form the divisor $2a+b$, that is, in the numerical case 160+ a quantity still unknown, but this quantity still unknown is also the quotient of the remainder 489 by the divisor. The process is to use 160 as a trial divisor, so as to get an idea what the unknown quantity may be. In this case we obtain 3, which added to 160 gives 163; and this multiplied by 3 gives 489. Hence the square root of 6889 is 83. Now in this mode of procedure a divisor quite distinct from the final result has to be formed. In the Soroban, however, whose peculiar feature in all operations is the disappearance of the various successive operations as the result is evolved, a distinct divisor does not appear. Thus, by an obvious transformation, we have

$$(a+b)^2 = a^2 + 2\left(a + \frac{b}{2}\right)b.$$

Comparing this as before with

$$6889^2 = 80^2 + 489$$

we see, that by *halving* the remainder 489, we may employ a itself, that is 80, as our trial divisor. In completing this step we must take $\frac{1}{2}b^2$ instead of b^2; and hence the importance in the Soroban method of the table of half squares. The simplicity of the method will be recognised from the following example. It is required to extract the square root of 418,609. As in ordinary ciphering, tick off the number in pairs, beginning at the right hand. Then clearly 600 is the first approximation to the value of the square root, or 6 is the first figure in the answer. Move up 6 on a convenient rod somewhat to the left. The successive operations are given symbolically below, the description following as in the previous examples.

	6	4	1	8	6	0	9
(a)			5	8	6	0	9
(b)			2	9	3	0	4.5
(c)	64			5	3	0	4.5
(d)				4	5	0	4.5
(e)					3	0	4.5
(f)						2	4.5
(g)	647						0

(a) Subtract 6^2 or 36 from 41, leaving 5.
(b) Halve the whole remainder 58,609.
(c) Use 6 as trial divisor of 29. This gives 4. Subtract 4 x 6 or 24 from 29, leaving 5, and consider 64 as the full divisor.
(d) Subtract half the square of 4 from 53. This completes the second stage.
(e) Start with 6 again as trial divisor of 45, or more accurately 600 as trial divisor of 4504.5. This gives 7. Subtract 7 x 6 or 42 from 45.
(f) Subtract 7 times 40 from the remainder 304.5.
(g) Subtract half the square of 7 from the remainder 24.5. 647 thus appears as the last divisor and, as there is no remainder, it is the square root of 418,609.

The whole process may easily be proved by considering the expansion of the square of a polynomial. Take for example, the quadrinomial *(a + b + c + d)*

$$\begin{aligned}(a+b+c+d)^2 &= a^2+b^2 \quad +c^2 \quad +d^2 \\ &\quad +2ab+2bc+2cd \\ &\quad +2ac+2bd \\ &\qquad\quad +2ad \\ &= a^2+2\left[\left(a+\frac{b}{2}\right)b\right. \\ &\qquad +\left(a+b+\frac{c}{2}\right)c \\ &\qquad \left.+\left(a+b+c+\frac{d}{2}\right)d\right]\end{aligned}$$

EXTRACTION OF CUBE ROOT (*Kai ryu hō*)

The difference in the Soroban and ciphering processes arises from the same cause as in the case of square root. That is, instead of preparing a divisor, the Soroban worker prepares the dividend. The much greater complication in the case of the cube root necessitates an *undoing* of the processes of preparation at each successive stage—a mode of operation which was obviated in the case of square root by the use of the table of half-squares. The analogous table of "third cubes" would be excessively awkward in operating with, because of the decimal non-finiteness of the fractions of three. The operator is expected to know by heart the table of cubes, or *Sai jō ku ku*. As in the ordinary ciphering method, the Soroban method depends upon the expression for the cube of a binomial. Consider, for example, the number 12,167. The first operation is to tick off in *threes*, that is in groups of ten-cubed.

Now 12 lies between the cubes of 2 and 3. Hence 20 is the first approximation to the cube root of 12,167. We have

$$\begin{aligned} 12{,}167 &= 8000 + 4167 \\ &= 20^3 + 4167 \end{aligned}$$

Now comparing this with the expression

$$\begin{aligned} (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + (3a^2 + 3ab + b^2)\, b \end{aligned}$$

we see that we must form a divisor whose most important part is $3a^2$, that is, 3 x 400 or 1200. Using 1200 as trial divisor of 4167, we get 3, which corresponds to the *b* in the general expression. We now form the complete divisor by adding to 1200 the expression

$$\begin{aligned} 3ab + b^2 &= 3 \times 20 \times 3 + 3 \times 3 \\ &= 180 + 9 \\ &= 189 \end{aligned}$$

Thus we find as final divisor 1389, which multiplied by 3, gives 4167; and hence 23 is the answer required.

The method on the Soroban depends upon the following transformation of the binomial expression

$$(a+b)^3 = a^3 + 3a\left(a + b + \frac{b^2}{3a}\right)b$$

Here, by dividing the remainder (after subtracting the cube of the first member) by that member and by 3, we obtain an expression whose principal part is *ab*, that is, the product of the first member and the as yet unknown second member. Hence, using *a* as trial divisor of the first figures of the prepared dividend we get *b*. In the process the *a* or first member of the answer is set down in such a position relatively to the original expression that the b when it is finally evolved falls into its proper place succeeding a. We now subtract b^2 from its proper place in the remainder; and the final remainder obtained is $b^3/3a$. Operating upon this by multiplying first by 3 and then by a, that is, by an exact reversal of the original process of preparation, we get b^3 left. We shall illustrate the process by extracting the root of 12,167 according to the Soroban method. The number is first ticked off by threes in the usual way, and the first number of the answer is set down on the first rod to the left of the highest triplet. In this particular example there are only two significant figures in the highest triplet, so that the 2 is set down two rods to the left of the first figure in the original number. The successive steps are as follows; and as position is of supreme importance in this operation, we shall symbolise the Soroban rods by ruled columns:—

a.	2		1	2	1	6	7
b.	2			4	1	6	7
c.	2		2	0	8	3	1
d.	2		6	9	4	3	2
e.	2	3	0	9	4	3	2
f.	2	3			4	3	2
g.	2	3			1	3	1
h.	2	3				2	7
i.	2	3				0	0

(a) Tick off into powers of 10^3 and consider the significant figures in the highest triplet, in this case 12. Two rods to the left set down 2, the highest integer whose cube (8) is less than 12.

(b) Subtract 2^3 or 8 from 12; or, to be more precise, subtract 20^3 or 8000 from the original number.

(c) Divide the remainder by the 2, which is the first found member of the answer. This, in accordance with the Soroban method of division, requires the first figure of quotient to be set down one rod to the left. Also it must be noted that the last *unit* is a fractional remainder and means really one-half.

(d) Divide by 3, carrying out the process until the last rod with the ½ remainder is reached. To this unit the unit of the fraction one-third which appears as a final remainder is added; so that the 2 on the last rod really means one-half and one-third. The division by 3 might be stopped at the preceding rod, so that instead of 69,432 we should have 69,411, in which the first unit means ⅓ and the second ½. There is greater chance of confusion, however, in this method than in the one shown, as will be seen when we come to the later stages.

(e) Divide by 2, but stop when the first figure in the quotient, in this case 3, is obtained.

(f) Continue this operation of division, regarding the newly obtained 3 as part of the divisor; or, in other words, subtract 3^2 or 9 from the next place to the right. We have now left a remainder represented by 43 and ½ and ⅓. This remainder is of the form b^3/a^3; and to bring it back to a workable form we must multiply it by $3a$. We must be careful, however, to do this so as to take proper account of the peculiar mixed fraction represented by 2 on the last rod to the right. The next two stages effect this.

(g) Multiply by 3, beginning however, at the second last rod, and thus undoing the operation *d*. Multiplication on the Soroban is accompanied by displacement to the right. Hence the product 3 x 43 or 129 has its last right-hand figure added to the rod containing the mixed remainder 2; and the final result of this operation gives 131, in which the last unit means as before one-half.

(h) Multiply by 2, beginning with the second last rod, and thus undoing the effect of operation *c*. The product 2 x 13 or 26 is added to the 1, and the 27 appears as the final expression.

(i) Subtract 3^3 or 27, and the remainder is zero.

Had we stopped in the operation *d* at an earlier point as suggested, we should have had to modify the reverse operation *g*. Thus, only the 4 of 411 would need to be multiplied by 3, giving of course 12 to be added to the first of the two units. The final result would have been of course 131, as already obtained.

The processes for extracting square root and cube root, on the other hand, imply a knowledge of mechanics much wider than the Abacus itself could ever teach. Square Root might perhaps have been evolved as a purely arithmetical operation on the Abacus; but Cube Root certainly could not. It seems more reasonable to suppose that both processes were deduced by some more general mathematical method, either algebraic or geometric.

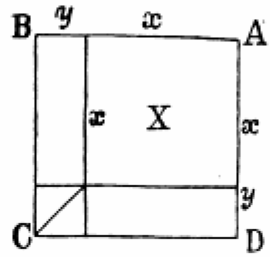

FIG. 12.

The geometrical aspect is indeed most instructive. Consider, for example, the square ABCD, from which has been subtracted the small square X, whose side *x* is known in finite terms. The L-shaped portion measures the remainder after X has been subtracted from the large square. From this remainder we have to find the length *y*, which with *x* makes up the side of the large square. The line drawn from C to the contiguous corner of X evidently cuts the L-shaped remainder into two halves. And each half in made up of the product of *x* and *y* and half the square of *y*. Here we have at once the suggestion of the Abacus rule for extracting square root. A similar consideration of the properties of the cube would lead to the Abacus rule for extracting the cube root. It is not probable, however, that these rules were discovered in this way. They are rather to be regarded as having been deduced from general algebraic considerations, just as our own rules are. They involve a knowledge of the binomial theorem, not necessarily in its complete generality, but so far at least as positive integers are concerned. It is known, however, that Chinese mathematicians have been acquainted for centuries with the binomial theorem, which they employed in the solution of equation of high degree. Hence it is almost certain that the Abacus rule for cube root is a formula deduced from the algebraic mode of solving such an equation as

$$x^3 - a = 0$$

The rule, of course, had to be formulated so as to suit the peculiar conditions of the arithmetic Abacus. The discussion of what might be called the algebraic Abacus or chessboard-like arrangement for solving equations is beyond the scope of the present paper.

See in this connection *A History of Japanese Mathematics*, by David Eugene Smith and Yoshio Mikami (Chicago, 1914).

EXHIBITS

1. Japanese Abacus. Lent by Cargill G. Knott, D.Sc.
2. Chinese Abacus. Lent by Major W. F. Harvey, I.M.S.

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