A short guide to the 5th lower bead

Revised March 2021

Introduction

It is a mystery why traditional Chinese and Japanese abacuses had five beads in their lower deck as only four are required from the point of view of decimal numbers representation. As no extant ancient document seems to explain it, this mystery will probably last forever and we are limited to conjectures to try to understand its origin. In this line, we could think that, when they first appeared, fixed beads abacuses were conceived in image and likeness of counting rods, from which they were called to inherit every algorithm. With counting rods, the use of five rods to represent number five was compulsory in order to avoid the ambiguity between one and five, at least initially, when neither a representation for zero nor a checkerboard a la Japanese sangi to sanban were used. Furnishing the abacus with five lower beads allowed a parallel or similar manipulations of beads and rods, bringing some kind of hardware and software compatibility to fixed beads abacuses, in fact, the first Chinese books on suanpan also dealt with counting rods, so that both instruments were learned at the same time. We could also invoke a certain desire for compatibility between the abacus and rods numerals that, in one way or another, have been in use until modern times. So, for instance, one would like to change all 5's to be represented by the five lower beads before writing down a result using rods numerals, in order to avoid very silly and catastrophic transcription mistakes.

Counting rods, by the way the most versatile and powerful abacus ever, had a flaw: it is extremely slow to manipulate. It is not a surprise that ancient Chinese mathematicians invented the multiplication table to speed up multiplication and that they also discovered the use of this multiplication table to also speed up division. Nor is it a surprise that they also discovered that, by using the abacus fifth bead, addition and subtraction operations could be somewhat simplified. They really had to be very sensitive to slowness.

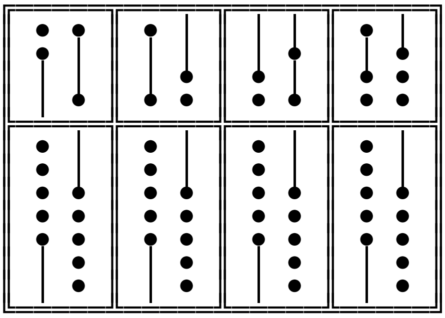
Yifu Chen, as part of his doctoral thesis[1], has systematically analyzed 16 classic works on the abacus: twelve Chinese books from the late Ming and Qing dynasties and four Japanese books from the Edo period in which addition and subtraction are studied. As a result, Chen finds four different modes of using the fifth bead in addition and two in subtraction. These modes range from intensive or systematic use of the fifth bead to sporadic use or no use at all. From all those texts, only one Chinese book from the late Ming dynasty make full use of the fifth bead, belonging to Chen's Mode 1 in both addition and subtraction: *Computational Methods with the Beads in a Tray* (*Pánzhū Suànfǎ* 盤珠算法) by Xú Xīnlǔ 徐心魯 (1573), by the way, the oldest extant book entirely devoted to the abacus. This fact should not lead us to the erroneous conclusion that the use of the fifth bead was a rarity of old times, since the books analyzed are neither treatises nor compendia on state-of-the-art abacus computation, but introductory manuals or textbooks for learning its use. Rather, it seems that the use or not of the fifth bead in these books corresponds more precisely to the didactic objective pursued by the authors, and that only Xu Xinlu considered it interesting to demonstrate it thoroughly from the beginning and included it in the syllabus of his course. It is mainly thanks to this work that we can rescue the traditional use of the fifth bead to simplify operations.

In what follows a small set of rules for the use of the fifth bead is presented along with their rationale and scope of use. These rules are not explicitly stated in any of the classical works, but can be inferred from the addition and subtraction demonstrations present in them, (especially in the *Panzhu Suanfa*) as is done in Chen's thesis

[1] doctoral dissertation defended in the Paris Diderot University (University Paris 7) in 2013.

Some terms and notation

(**F**) to denote a *lower five* (five lower beads set) as opposed to:
(**5**) *upper five* (one upper bead set).
(**T**) ten on a rod (one upper bead and five lower beads set). On the suanpan, it is also a *lower ten* as opposed to (**t**) an *upper ten* (two upper beads set).
(**Q**) *lower fifteen* on a rod (two upper beads and five lower beads set) as opposed to (**q**) *upper fifteen* (suspended upper bead on the 2:5, three upper beads set on the 3:5).
(**carry**) this represents number 1 when it is to be added to a column as a carry from the right.



F 5 T t Q q Q q

Rules for addition

- **a1**) Never use the 5th bead in addition except in the two cases that follow.
- **a2**) **`4 + carry = F`**
- **a3**) **`9 + carry = T`**[2]

The rationale behind

Rule **a1** goal is simply to always leave an unused lower bead at our disposal in case the current column has to accept a future carry from the right, while rules **a2** and **a3** specify the use of the 5th bead in such a situation. Then, we can expect to obtain:

- a reduced number of finger movements because we avoid to deal with both upper and lower beads
- to avoid skipping rods and to reduce the left-right hand displacement span
- to avoid any "carry run" to the left (think of `99999+1=999T0` instead of `99999+1=100000`)

The advantage

The above advantages are automatically realized by using rules **a2** and **a3**, but rule **a1** is of a different nature. Rule **a1** is a provision for the future, It will simplify things if a future carry actually falls on the current column (which happens about 50% of the time on average), but it will simplify nothing otherwise. Rule **a1** is so a kind of a bet (subtraction rules below are also of the same nature).

The scope of use

Rules **a1** to **a3** are for columns that can receive a carry, which excludes the rightmost column in normal (rightward) operation.

In inverse (leftward) operation, no column will receive a future carry from the right, so that rule **a1** is out of scope and does not operate, but rules **a2** and **a3** should always be used. (I mention this because an ancient technique, now defunct, used leftward operation in alternation with normal operation to avoid long hand displacements. Not of general use but an extremely interesting exercise anyway).

Exceptionally, if you do know that some column will never receive a carry, you are also free of rule **a1**. (This seems a strange situation, but I need to introduce it to cope with the central part of the Test Drive below).

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[2] You can see the above addition rules mentioned in a slightly different way by Yifu Chen in the chapter: *The Education of Abacus Addition in China and Japan Prior to the Early 20th Century* of the book: *Computations and Computing Devices in Mathematics Education Before the Advent of Electronic Calculators*. Springer 2018 https://doi.org/10.1007/978-3-319-73396-8

Rules for subtraction

- **s1**) Always use lower fives (**F**) instead of upper fives (**5**).
- **s2**) Never leave a cleared rod (**0**) if you can borrow from the adjacent left rod (but not from a farther one!), leave a **T** instead (i.e. `27-7 = 1T` should be preferred to `27-7 = 20`).

Remark: These two rules do not apply on rods where you are borrowing from, i.e. `112-7 = 10F` (not `TF`) and `62-7 = 5F` (not `FF`).

The rationale behind

Both rules tend to leave activated lower beads at our disposal for the case we need to borrow from them in the future (it is like always holding small change in our pocket just in case), saving us some movements and/or wider or more complex hand displacements, such as borrowing from non-adjacent columns or skipping rods.

The advantage

Is not automatically obtained, it is only fulfilled when we actually need to borrow from the present rod. This is similar to the case of addition rule **a1**.

The scope of use

Once more, the rightmost column is outside the scope of these rules as we will never borrow from it.

Also, In leftward or inverse operation we will never borrow from the current column, so these rules do not apply (which may be seen as an additional reason to prefer rightward operation in normal use).

Test drive

It was common in ancient books on the abacus to demonstrate addition and subtraction using the well-known exercise that consists of adding the number `123456789` nine times to a cleared abacus until the number `1111111101` is reached, and then erase it again by subtracting the same number nine times[3]. You can find the sequence of intermediate results of the *Panzhu Suanfa* in this 1982 article by Hisao Suzuki (鈴木 久男): *Chuugoku ni okeru shuzan kagen-hou* 中国における珠算加減法 (*Abacus addition and subtraction methods in China)*[4]. This is a Japanese text (spiced up with some classical Chinese) that deals with addition and subtraction methods as they appear in various Chinese books from the 16th century. In pages 12-17, the *Panzhu Suanfa* version of the `123456789` exercise is graphically displayed on the upper series of 1:5 diagrams[5]. The short Chinese phrases below each bar specify how the current digit was obtained (**Table 1** in the Appendix A below serves a similar purpose but in a different and more convenient way for us).

Using the addition rules explained above, we should get the following sequence of results each time we complete the addition of `123456789` (see **Table 1** for more details):

```
000000000, 123456789, 246913F78, 36T36T367, 4938271F6,
617283945, 74073T734, 864197F23, 9876F4312,    ...
```

3 This exercise seems to have the Chinese name : *Jiǔ pán qīng* (九盤清), meaning something like "clearing the nine trays" (I guess, but I am unsure). I wonder if it also has a Japanese name.
4 Click on the pdf icon or alternatively get the file here.
5 For convenience they are reproduced in Appendix B

at this point, adding `123456789` once more results in `1111111101`, but this number appears in the *Panzhu Suanfa as:*

TTTTTTTT1

which cannot be obtained by the use of the above rules only. A similar situation occurs when repeating this exercise but starting with `999999999` instead of a cleared abacus (see **Table 2**), reaching `1TTTTTTTT0`. This is why I introduced the last comment on the scope of addition rules above. It might be that, by inspection or intuition, we realize that using the 5th bead here does not generate any carry, so that we can overcome the **a1** rule and proceed to this, somewhat theatrical, result ...

From here, by subtraction we should get:

TTTTTTTT1, 9876F4312, 864197523, 740740734, 61728394F,
493827156, 36T370367, 246913578, 123456789, 000000000

As it can be seen here, few `F`'s and `T`'s appear on the intermediate results, but a few more appear in the middle of calculation (**Table 1**), being immediately converted to `4`'s and `9`'s by borrowing, which is the purpose for which they were introduced. The `F`'s and `T`'s remaining on the intermediate results are only the *unused* ones.

Additional rules

Of course, the rules for addition can also be directly used in multiplication and the rules for subtraction in division, roots, etc.

Additionally, if using *kijoho* 帰除法 (Chinese or traditional division method) on the 2:5 or 3:5 abacus, we can introduce an additional rule:

- **k1**) Always use lower five's, ten's, and fifteen's (**F**, **T**, **Q**) when adding to the remainder after application of the Chinese division rules.

This is so because, although we are adding to a rod, the next thing we will do is start subtracting from it (if the divisor has more than one digit). It is a kind of extension of the first rule for subtraction (**s1**).

By the way, you may sometimes find somewhat conflicting the use of the second rule for subtraction (**s2**) in Chinese division. For instance, 1167/32 = 36,46875

```
abcdefg
-------
32 1167 1/3->3+1 rule
32 3267 -3*2=-6 in f, use 2nd subtraction rule
    -6
32 31T7
```

Now, which rule should be used here? 1/3->3+1 or 2/3->6+2 ? In fact, we can use any of them and revise up as needed, but it is faster to realize that the remainder is actually 3207 so that the second Chinese rule is the appropriate one, so, simply change columns `ef` to 62 and continue.

```
abcdefg
-------
32 3627
...
```

Finally, if you are using the traditional Chinese multiplication method or similar on the *suanpan*, you may face overflow on some columns, so that an additional rule:

- **m1**) **`[14] + carry = Q`**

can also be considered (but I am not aware of having ever used it).

About the advantage

As Hannu and Totton pointed out, the use of the 5th bead may reduce the number of bead or finger movements required in some calculations, and the detailed example offered by them shows how it is achieved. Some time ago, I estimated this reduction in roughly 10% on the average, based on the `123456789` exercise and some of its derivatives. Nevertheless, I could have used different criteria[6] to count movements so I don't value this numeric estimate too much.

As I see things, the advantage of the 5th bead goes a little beyond simply reducing the number of fingers movements, as it also reduces the number and/or the extent of other hand gestures required in calculations (hand displacement, changes of direction, skipping rods,...). It seems pretty obvious that each gesture,

- as a physical process, takes a time to complete
- as governed by our brains, requires our attention, consuming (mental or biochemical) energy
- as we are not machines but humans, has a chance to be done in the wrong way, introducing mistakes[7]

So, under this optics, we can expect that the use of the 5th bead will result in a somewhat **faster**, more **relaxed** and **reliable** calculation by reducing the total number of required gestures. It is not easy to measure this triple advantage using a single parameter.

Skipping columns, as Yifu Chen comments in his two works mentioned above, seems to have traditionally been viewed as something to be avoided as a possible source of errors. Without this concept the subtraction rule (**s2**) cannot be understood since it does not always lead to a reduction in the number of finger movements, but it always reduces the range of hand movement and the need to skip rods. I personally attach great importance to this point based on my own experience. I still fear divisors and roots with embedded zeros and if I became an enthusiast of traditional methods it is solely because I drastically reduced my error rate in division and multiplication as soon as I started using the more compact traditional arrangements used for both operations. No matter how hard and for how long I practiced modern methods, I could never change this.

In any case, the advantage of using the fifth bead, although not negligible, is only modest, and each one must decide whether it is worth using it or not. For me the best test of the efficiency of the fifth bead is to use a 1:4 and feel the extra work required to complete the calculations on it.

6 Some movements can be done simultaneously, should they be counted independently?

7 From a mathematical point of view, this makes bead arithmetic a game similar to Russian Roulette!

Appendix A

Explanation

Two tables are included here.

Table 1 is an step by step or digit by digit realization of the `123456789` exercise, so that it is possible to see in detail how the intermediate results are obtained (except for the order of movement of the fingers that depend on personal tastes, i.e. use of sakidama or atodama for each operation). I think this is especially important for subtraction where the `F`'s and `T`'s can disappear as mentioned above. Please note that these are my own results, and not from any ancient book!

Table 2 shows my own intermediate results for the `123456789` exercise repeated over a background of `111111111`, `222222222`, `333333333`, ...etc. I add this as a guide so that you have additional results to practice with. I use exercises like this every day.

In any case, don't pay too much attention to the rightmost digit, as it is outside the scope of use of the 5th bead as explained in the text.

Table 1: The 123456789 exercise step by step

ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI	
---------		---------		---------		---------		---------	
000000000		123456789		246913F78		36T36T367		4938271F6	
100000000	A+1	223456789	A+1	346913F78	A+1	46T36T367	A+1	5938271F6	A+1
120000000	B+2	243456789	B+2	366913F78	B+2	48T36T367	B+2	6138271F6	B+2
123000000	C+3	246456789	C+3	369913F78	C+3	49336T367	C+3	6168271F6	C+3
123400000	D+4	246856789	D+4	36T313F78	D+4	49376T367	D+4	6172271F6	D+4
123450000	E+5	246906789	E+5	36T363F78	E+5	49381T367	E+5	6172771F6	E+5
123456000	F+6	246912789	F+6	36T369F78	F+6	493826367	F+6	6172831F6	F+6
123456700	G+7	246913489	G+7	36T36T278	G+7	493827067	G+7	6172838F6	G+7
123456780	H+8	246913F69	H+8	36T36T358	H+8	493827147	H+8	617283936	H+8
123456789	I+9	246913F78	I+9	36T36T367	I+9	4938271F6	I+9	617283945	I+9

ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI	
---------		---------		---------		---------	
617283945		74073T734		864197F23		9876F4312	
717283945	A+1	84073T734	A+1	964197F23	A+1	T876F4312	A+1
737283945	B+2	86073T734	B+2	984197F23	B+2	TT76F4312	B+2
740283945	C+3	86373T734	C+3	987197F23	C+3	TTT6F4312	C+3
740683945	D+4	86413T734	D+4	987597F23	D+4	TTTTF4312	D+4
740733945	E+5	86418T734	E+5	987647F23	E+5	TTTTT4312	E+5
740739945	F+6	864196734	F+6	9876F3F23	F+6	TTTTTT312	F+6
74073T645	G+7	864197434	G+7	9876F4223	G+7	TTTTTTT12	G+7
74073T725	H+8	864197F14	H+8	9876F4303	H+8	TTTTTTT92	H+8
74073T734	I+9	864197F23	I+9	9876F4312	I+9	TTTTTTTT1	I+9

ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI	
---------		---------		---------		---------		---------	
TTTTTTTT1		9876F4312		864197523		740740734		61728394F	
9TTTTTTT1	A-1	8876F4312	A-1	764197523	A-1	640740734	A-1	F1728394F	A-1
98TTTTTT1	B-2	8676F4312	B-2	744197523	B-2	620740734	B-2	49728394F	B-2
987TTTTT1	C-3	8646F4312	C-3	741197523	C-3	617740734	C-3	49428394F	C-3
9876TTTT1	D-4	8642F4312	D-4	740797523	D-4	617340734	D-4	49388394F	D-4
9876FTTT1	E-5	8641T4312	E-5	740747523	E-5	617290734	E-5	49383394F	E-5
9876F4TT1	F-6	864198312	F-6	740741523	F-6	617284734	F-6	49382794F	F-6
9876F43T1	G-7	864197612	G-7	740740823	G-7	617283T34	G-7	49382724F	G-7
9876F4321	H-8	864197532	H-8	740740743	H-8	6172839F4	H-8	49382716F	H-8
9876F4312	I-9	864197523	I-9	740740734	I-9	61728394F	I-9	493827156	I-9

ABCDEFGHI		ABCDEFGHI		ABCDEFGHI		ABCDEFGHI	
---------		---------		---------		---------	
493827156		36T370367		246913578		123456789	
393827156	A-1	26T370367	A-1	146913578	A-1	023456789	A-1
373827156	B-2	24T370367	B-2	126913578	B-2	003456789	B-2
36T827156	C-3	247370367	C-3	123913578	C-3	000456789	C-3
36T427156	D-4	246970367	D-4	123F13578	D-4	000056789	D-4
36T377156	E-5	246920367	E-5	123463578	E-5	000006789	E-5
36T371156	F-6	246914367	F-6	123457578	F-6	000000789	F-6
36T370456	G-7	246913667	G-7	123456878	G-7	000000089	G-7
36T370376	H-8	246913587	H-8	123456798	H-8	000000009	H-8
36T370367	I-9	246913578	I-9	123456789	I-9	000000000	I-9

Table 2: The 123456789 exercise over a background

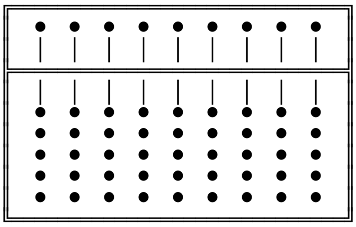
0	1	2	3	4
000000000	0111111111	0222222222	0333333333	0444444444
123456789	02345678T0	0345678T11	045678T122	05678T1233
246913F78	0357T24689	046913F7T0	057T246911	0691357T22
36T36T367	0481481478	0592592F89	06T36T36T0	0814814811
4938271F6	0604938267	0715T49378	082715T489	09392715T0
617283945	0728394TF6	08394T6167	09F0617278	1061738389
74073T734	08F18F1845	09629629F6	1074073T67	118F18F178
864197F23	097F308634	1086419745	1197F2T8F6	1308641967
9876F4312	109876F423	1209876F34	1320987645	14320987F6
TTTTTTTT1	1222222212	1333333323	1444444434	1555FFFF45
9876F4312	1098765423	1209876534	132098764F	1432098756
864197523	097F308634	108641974F	1197F30856	1308641967
740740734	08F18F184F	0962962956	0T74074067	118F18F178
61728394F	072839F056	0839F06167	09F0617278	0T61728389
493827156	05T4938267	0716049378	0827160489	093827159T
36T370367	0481481478	0592592589	06T370369T	0814814811
246913578	0357T24689	046913579T	0F7T246911	0691358022
123456789	023456789T	0345678T11	04F678T122	0F678T1233
000000000	0111111111	0222222222	0333333333	0444444444

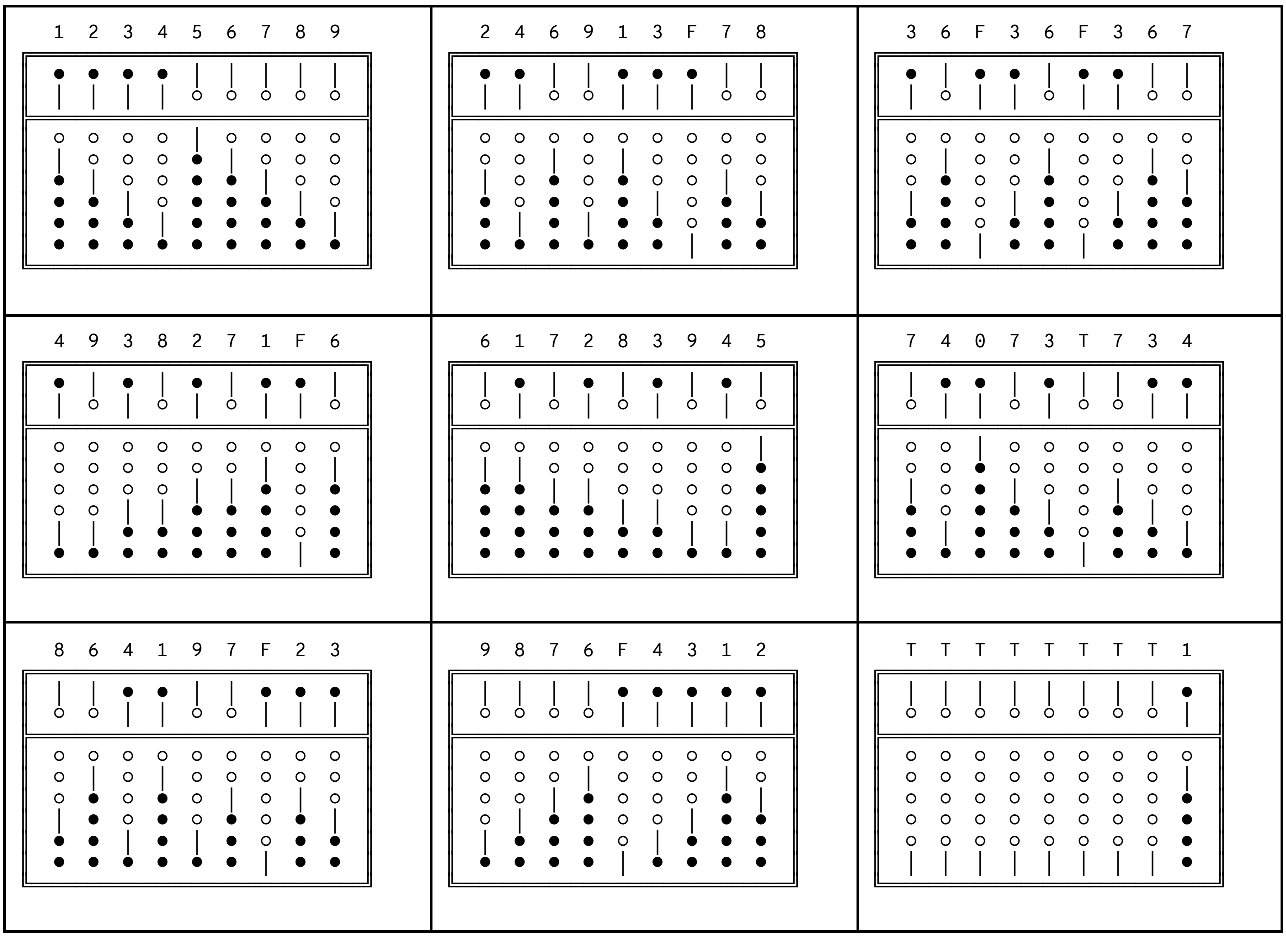
5	6	7	8	9
0555555555	0666666666	0777777777	0888888888	0999999999
0678T12344	078T1234F5	08T1234F66	0T1234F677	11234F6788
07T2469133	091357T244	0T246913F5	11357T2466	1246913F77
0925925922	1036T36T33	1148148144	12592592F5	136T36T366
1049382711	115T493822	12715T4933	1382715T44	14938271F5
11728394T0	128394T611	1394T61722	1F06172833	1617283944
1296296289	14073T73T0	1F18F18F11	1629629622	174073T733
14197F2T78	1530864189	164197F2T0	17F3086411	1864197F22
1543209867	1654320978	176F431T89	1876F431T0	19876F4311
16666666F6	1777777767	1888888878	1999999989	1TTTTTTTT0
1F43209867	16F4320978	176F432089	1876F4319T	19876F4311
14197F3078	1F30864189	164197529T	17F3086411	1864197522
1296296289	140740739T	1F18F18F11	1629629622	1740740733
117283949T	12839F0611	139F061722	14T6172833	1617283944
0T49382711	115T493822	1271604933	1382716044	149382715F
0925925922	0T36T37033	1148148144	125925925F	136T370366
07T2469133	0913580244	0T2469135F	11357T2466	1246913577
0678T12344	078T12345F	08T1234566	0T12345677	1123456788
0FFF55555F	0666666666	0777777777	0888888888	0999999999

Appendix B

Diagrams[8] of exercise 123456789 as they appear in the *Panzhu Suanfa* (corrected): Addition

0 0 0 0 0 0 0 0 0





[8] Prepared with txt-abacus (https://github.com/jccsvq/txt-abacus).

Diagrams of exercise 123456789 as they appear in the *Panzhu Suanfa* (corrected): Subtraction