**Balanced Ternary Arithmetic on the Abacus**

What is balanced ternary? It is a ternary number system where the digits of a number are powers of three rather than powers of ten, but instead of adding multiples of one or two times various powers of three to make up the desired number, in balanced ternary some powers of three are added and others are subtracted. Each power of three comprisising the number is represented by either a +, a -, or 0, to show that the power of three is either added, subtracted, or not present in the number. There are no multiples of a power of three greater than +1 or -1, which greatly simplifies multiplication and division. Another advantage of balanced ternary is that all integers can be represented, both positive and negative, without the need for a separate symbol to indicate plus or minus. The most significant ternary digit (trit) of any positive balanced ternary number is + and the most significant trit of any negative number is -. Here is a list of a few numbers in decimal, ternary, and balanced ternary:

| Decimal | Ternary | Balanced ternary | |
|---|---|---|---|
| -6 | - 20 | - + 0 | (-9+3) |
| -5 | - 12 | - + + | (-9+3+1) |
| -4 | - 11 | - - | ( -3-1) |
| -3 | - 10 | - 0 | etc. |
| -2 | - 2 | - + | |
| -1 | - 1 | - | |
| 0 | 0 | 0 | |
| 1 | 1 | + | |
| 2 | 2 | + - | |
| 3 | 10 | + 0 | |
| 4 | 11 | + + | |
| 5 | 12 | + - - | |
| 6 | 20 | + - 0 | |

Notice that the negative of a balanced ternary number is formed simply by inverting all the + and - signs in the number.

Thomas Fowler in England, about 1840, built a balanced ternary calculating machine capable of multiplying and dividing balanced ternary numbers. For more information, see: http://www.mortati.com/glusker/fowler/about.htm
Some experimental balanced ternary computers have been built in Russia.

For manual calculations in balanced ternary, it turns out that an abacus is quite useful. Two beads on each rod are all that are required, but of course a standard suan-pan or soroban may be used with all beads ignored except the “heaven” bead and the “earth” bead nearest the counter beam. My preference is to use the “heaven” bead to represent minus, and the "earth" bead to represent plus, so that we can add by moving beads up (away from the user) and subtract by moving beads down (toward the user). A minus is represented by the “heaven” bead next to the counter beam and a plus by the “earth” bead next to the counter beam. Either of the other two possibilities may be used for zero, but I prefer to represent zero by having both beads next to the counter beam. The normal “cleared” state, with both beads away from the counter is reserved for unused rods and as a marker to separate the multiplier and product during multiplication, or the quotient and dividend during division. We are able to use such a "marker" in balanced ternary, because, unlike in decimal arithmetic, we have a bead configuration which is not used; i.e., doesn't represent +, -, or 0.

In the following diagrams, the "heaven" bead is on top and the "earth" bead is below, with the horizontal line representing the counter beam; the rods are not shown. The arithmetic works as follows:

```
 - and +  → 0        0 and +  → +         + and -  → 0         0 and -  → -
                               *           *
  *         *         *                                *         *          *
 ___       ___       ___      ___         ___         ___       ___        ___
      →     *         *   →    *           *    →      *         *    →
  *                                                                         *
```

and adding 0 to + or – or 0 does not change anything.

As you can see, we add by moving a bead up, and subtract by moving a bead down. There are two special cases, which generate a carry to the next rod (column) to the left. These are "+ and +", and "– and -". "+ and +" is equal to – with a + carry to the rod to the left; "- and -" is equal to + with a – carry to the rod to the left. We can see from the diagrams above, that if we have a + and try to add a +, that we cannot move a bead up, so we move both beads down and carry a + to the rod to the left (push a bead up on that rod). Similarly, if we have a – and try to add a -, we are unable to move a bead down, so must move both beads up and carry a – to the rod to the left(pull a bead down).. These rules, combined with the fact that multiplying a number by – simply changes all + to – and – to + in that number, are all we need to multiply balanced ternary numbers on the (two bead) abacus.

**To add + to a rod, push a bead up; if you can't, then pull both beads down and push up a bead on the rod to the left. To add - to a rod, pull a bead down; if you can't, then push both beads up and pull down a bead on the rod to the left.**

One useful tip to avoid losing your place if you have carries across more than one rod is to always keep one finger on the bottom of the abacus right below the product rod which is currently being worked on. This way, if you have to back up to propagate a carry across multiple rods, you can easily pick up where you left off.

Here's an example to illustrate how multiplication works. I prefer to set the multiplier on the abacus and develop the product to the right of the multiplier, with a single marker rod separating the two; this marker rod has both beads away from the counter beam. The marker rod will be moved to the left as the product is developed leftward and the multiplier whittled down from right to left. Some may prefer not to use a marker rod, or to use more separation between the multiplier and product - the method described is just one way to do this. The example is:

```
     789                        + 0 + - + - 0         For reference: + multiplicand is   + 0 + - + - 0
x    456      or          x     + -  0 - 0 – 0                       -  multiplicand is  – 0 - + - + 0
----------          --------------------------
359,784             + - 0 0 + 0 - - - - + 0 0
```

Clear the abacus (all beads away from the counter beam) and set the multiplier:

```
+   -   0   -   0   -   0   M           (note: M = the separation marker rod)

*                           *   *   *   *   *   *   *   *
    *   *   *   *   *   *
----------------------------------------------------------
*       *       *       *
    *       *       *       *   *   *   *   *   *   *   *
A   B   C   D   E   F   G   H   I   J   K   L   M   N   O
```

Multiply the multiplicand by the rightmost digit of the multiplier (0) from rod G and set this value, 0000000, to the right of the marker on rods I to O, then zero the marker rod H and replace the rightmost digit of the multiplier (rod G) with the new marker.

```
+   -   0   -   0   -   M   0   0   0   0   0   0   0   0

*                       *
    *   *   *   *   *       *   *   *   *   *   *   *   *
----------------------------------------------------------
*       *       *           *   *   *   *   *   *   *   *
    *       *       *   *
A   B   C   D   E   F   G   H   I   J   K   L   M   N   O
```

Multiply the multiplicand by the rightmost digit of the multiplier (-) from rod F and add this value, -0-+-+0, to the previous product, starting on rod H, just to the right of the marker, then zero the marker rod and replace the rightmost digit of the multiplier with the marker. Always work from left to right starting at the marker; propagate carries as needed as you come to them, then continue on.

```
+   -   0   -   0   M   0   -   0   -   +   -   +   0   0
*                   *                   *       *
    *   *   *   *       *   *   *   *       *       *   *
----------------------------------------------------------
*       *       *       *       *       *       *   *   *
    *       *       *       *       *       *
A   B   C   D   E   F   G   H   I   J   K   L   M   N   O
```

Multiply the multiplicand by the rightmost digit of the multiplier (0) from rod E and add this value, 0000000, to the previous product, starting just to the right of the marker, then zero the marker rod and replace the rightmost digit of the multiplier with the marker.

```
+   -   0   -   M   0   0   -   0   -   +   -   +   0   0
*               *                       *       *
    *   *   *       *   *   *   *   *       *       *   *
----------------------------------------------------------
*       *           *   *       *       *       *   *   *
    *       *   *           *       *       *
A   B   C   D   E   F   G   H   I   J   K   L   M   N   O
```

Multiply the multiplicand by the rightmost digit of the multiplier (-) from rod D and add this value, -0-+-+0, to the previous product, starting just to the right of the marker.   There are several carries in

this operation, so I'll take this step by step:
Beginning with rod F, we must add a -, so pull a bead down. On rod G, we add a 0 so there is no change. On rod H, we must add a - but both beads are already pulled down, so we have to push both beads up and then pull down a bead on rod G. On rod I, we add a + so just push a bead up. For rod J, we need to add a - but can't because both beads are pulled down, so push both beads up and pull the top bead down in rod I. Things get a little more complicated at rod K because we will have a carry across two rods; we need to add a + to rod K but can't push a bead up, so pull both beads down and try to push a bead up on rod J, but both beads on J are already pushed up, so we pull both beads down and push up a bead on rod I. Rod L has a 0 added, so there is no change here. After zeroing the marker rod E and moving the marker to rod D, we end up with:

```
+ - 0 M 0 - - + + - - - + 0 0
*     *       * *       *
  * *   * * *     * * *   * *
-----------------------------
*   *   *     * *       * * *
  *   *   * *     * * *
A B C D E F G H I J K L M N O
```

Multiply the multiplicand by the rightmost digit of the multiplier (0) from rod C and add this value, 0000000, to the previous product, starting just to the right of the marker, then zero the marker rod and replace the rightmost digit of the multiplier with the marker.

```
+ - M 0 0 - - + + - - - + 0 0
*   *         * *       *
  *   * * * *     * * *   * *
-----------------------------
*     * *     * *       * * *
  * *     * *     * * *
A B C D E F G H I J K L M N O
```

Multiply the multiplicand by the rightmost digit of the multiplier (-) from rod B and add this value, -0-+-+0, to the previous product, starting just to the right of the marker Rod D adds -, so pull a bead down. Rod E adds 0, so no change. Rod F adds -, so push both beads up and pull down a bead on rod E. Rod G adds +, so push up the earth bead. Rod H adds -, so pull down the heaven bead. Rod I adds +, so pull down both beads and push up a bead on rod H. Rod J adds 0, so there is no change here. After zeroing the marker rod and moving the marker to rod B, we have:

```
+ M 0 - - + 0 + - - - - + 0 0
* *       *   *         *
    * * *   *   * * * *   * *
-----------------------------
*   *     * * *         * * *
  *   * *       * * * *
A B C D E F G H I J K L M N O
```

The final step is to multiply the multiplicand by the rightmost digit of the multiplier (+) from rod A and add this value, +0+-+-0, to the previous product, starting at rod C, just to the right of the marker. Rod C adds a +, so push up a bead. Rod D adds a 0, so no change. Rod E adds a +, so push up the lower bead. Rod F adds a -, pull down the upper bead. Rod G adds a +, so push up a bead. Rod H adds a -, pull down the upper bead. Rod I adds a 0, so no change. Now clear rod A (both beads away from the counter beam), and we have the complete product on rods C to O:

```
   M  +  -  0  0  +  0  -  -  -  -  +  0  0
*  *  *           *                 *
         *  *  *     *  *  *  *  *     *  *
---------------------------------------------
      *     *  *  *  *              *  *  *
*  *     *                 *  *  *  *
A  B  C  D  E  F  G  H  I  J  K  L  M  N  O
```

Obviously, this same technique could be used for division, with the quotient being developed from left to right on the left side of the marker as the dividend is whittled down from left to right on the right side of the marker. A + in the quotient would add the negative of the divisor to the dividend remainder, and a - would add the positive divisor. For example, here is 456 / 24 = 19 worked out in balanced ternary:

```
              + -  0 +
+ 0 - 0 | + - 0 - 0 - 0
          -  0 + 0
             - + - 0          a negative remainder is allowed in balanced ternary
             -  0 + 0
                  + 0 - 0
                  -  0 + 0
                         0
```

We could start this calculation with the marker on rod B, rod A = 0, and rods C - I = + - 0 - 0 - 0. The first quotient trit would be entered on rod A, the first subtraction completed, then rod B zeroed and the marker moved to rod C, etc. The quotient ends up on rods A - D.

A note about balanced ternary division - whenever a quotient trit is + or -, the corresponding addition or subtraction from the dividend or remainder must produce a new (complete) remainder whose absolute value is less than the absolute value of the initial (complete) remainder; otherwise the quotient trit is set to 0. For example, in 456/12 = 38:

```
            +
 + + 0 | + - 0 - 0 - 0        absolute value = 456
         - -  0
         - + 0 - 0 - 0        absolute value = 516, larger than 456, so this quotient trit is 0, not +
continuing:
          0  + +  + -             =  38
 + + 0 | + - 0 - 0 - 0        absolute value = 456
          -  - 0
           + - -  0 - 0       absolute value = 132, OK
             - -  0
               +  0 - 0       absolute value = 24, OK
               -  -  0
                  - -  0      absolute value = 12, OK
                  + + 0
                       0
```

For divisions with remainders, it is possible to end up with a negative remainder. This may be converted to a positive remainder by adding the divisor to it and adding a - to the quotient, or vice versa if you have a negative quotient and a positive remainder.

**Balanced Ternary to Decimal Conversion and Decimal to Balanced Ternary Conversion**

First, consider balanced ternary (B.T.) to decimal conversion. Assuming we don't have a table of powers of three at hand, the most obvious way to do this is to start at the most significant end of the B.T. number, which will be assumed to be a + (if it's a -, invert the number, convert it, and then affix a minus sign to the front of the decimal number). Start with 1 for the leading +, and multiply that by 3. Then add the next B.T. digit and multiply the sum by 3, and continue in this manner until the last digit is added (don't multiply by 3 after adding the last one). If we use the standard technique for multiplication on the abacus, the product will "walk" to the right as we do successive multiplications, but there is a technique for multipliying in place which keeps the least significant digit of the product fixed. (Thanks to Fernando Tejón - http://webhome.idirect.com/~totton/soroban/MultiFac/).

For the B.T. conversion, this method requires adding to each rod the double of the number on that rod. So, for example, if the number is 274, to triple it, we add 04 to 2, 14 to 7, and 8 to 4:

```
     2 7 4
  + 0 4
 --------------
     6 7 4
    + 1 4
 --------------
     8 1 4
     + 0 8
 -------------
     8 2 2
```

As an example, to convert the balanced ternary number +0+-+-0 to decimal, we start at the left end:

```
      0 0 0
 +        1   for the leading +
    --------
  +  0 0 1
  +    0 2   to multiply by 3
   ---------
     0 0 3
   +     0       0
   ----------
     0 0 3
   +   0 6    multiply by 3
   ---------
     0 0 9
   +     1       +
  ----------
     0 1 0
 +   0 2     multiply by 3
```

```
 ------------
     0 3 0
 -       1        -
------------
     0 2 9
+    0 4      multiply by 3
 ------------
     0 6 9
+       1 8
 -----------
     0 8 7
 +        1       +
 -----------
     0 8 8
 +  1 6       multiply by 3
 -----------
     2 4 8
  +     1 6
 -----------
     2 6 4
 -        1         -
  ----------
     2 6 3
+   0 4          multiply by 3
-------------
     6 6 3
+       1 2
 -------------
     7 8 3
+        0 6
 ------------
     7 8 9
  +        0         0
 ------------
     7 8 9       the decimal value of +0+-+-0
```

This may look long and complicated when written out this way, but is easily and quickly done on the abacus.

Now consider decimal to balanced ternary conversion. It seems easiest to me to first convert the decimal number to a standard (unbalanced) ternary number, then convert that to balanced ternary. The most obvious way to convert from decimal to ternary is successive division by 3, taking the remainder after each division as a ternary digit. If we do this on the abacus, using the standard abacus division method, the quotient will "walk" to the left with each division. To avoid this, I use the reverse of the method used above for multiplication by three. We work from right to left, subtracting 2/3 of the smallest multiple of three which can be formed from the digit on the rod being operated on, or that digit plus 10, or plus 20. The 2/3 value to subtract can be listed in a table in which the index values are the digit on the rod being operated on:

| Digit | Value to subtract | |
|---|---|---|
| 0 | 0 | (2/3 of 0) |
| 1 | 14 | (2/3 of 21) |
| 2 | 8 | (2/3 of 12) |
| 3 | 2 | (2/3 of 3) |
| 4 | 16 | (2/3 of 24) |
| 5 | 10 | (2/3 of 15) |
| 6 | 4 | (2/3 of 6) |
| 7 | 18 | (2/3 of 27) |
| 8 | 12 | (2/3 of 18) |
| 9 | 6 | (2/3 of 9) |

Two digit numbers in the second column are subtracted from the rod being operated on and the rod to its left.

The only complication is that the number must be evenly divisible by three; if it is not, we must first subtract either one or two to make it divisible by three, and the number we subtract is taken as a digit of the ternary number we are forming. Recall that a number can be tested for divisibility by three by summing the digits - if the sum of the digits is a multiple of three, then the number is divisible by three. This is an easy test on the abacus - just count the number of heaven beads on the counter beam, multiply by 5 and add the digits modulo three to form a single digit number, then start adding the number of earth beads to that, modulo three.

As an example, to convert 456 to ternary:

Sum of the digits of 456 is 6 (= 2*5 or 10 = 1 plus 5 earth beads = 6), so it is divisible by 3 and the first (least significant) ternary digit is **0**
Now, using the table above, divide by 3 by subtracting 2/3 of the number in each column.

```
      4 5 6        subtract 4 from 6
  -     0 4
   ----------
      4 5 2        subtract 10 from 5
  -   1 0
  ---------
      3 5 2        subtract 2 from 3
 -   2
  ----------
      1 5 2
```

Sum of the digits of 152 is 8, so we must subtract **2** which becomes the next ternary digit

```
      1 5 2
  -       2
     -------
      1 5 0        now divide by 3:
 -    1 0          subtract 10 from 5
    --------
        5 0
```

Sum of the digits of 50 is 5, not divisible by 3, so subtract **2** which is the next ternary digit

```
        5 0
```

```
  -    2
      -----
       4 8        now divide by 3:
    -  1 2        subtract 12 from 8
     ------
       3 6        subtract 2 from 3
  -  0 2
    ---------
       1 6
```

Sum of the digits is 7, not divisible by 3, so subtract **1**, which is the next ternary digit

```
       1 6
  -      1
      -----
       1 5        now divide by 3:
    -  1 0        subtract 10 from 5
     ------
         5
```

Sum of the digits is 5, so subtract **2** which is the next ternary digit

```
         5
  -      2
       -----
         3        now divide by 3:
    -  0 2        subtract 2 from 3
    --------
         1
```

Sum of the digits is 1, which is not divisible by three, so subtract **1** which is the next ternary digit

```
         1
    -    1
    -------
         0          finished - the complete ternary number is 121220
```

One method for converting from ternary to balanced ternary is to first, add 11111...11 to the ternary number with carries propagated, then subtract the same number with no borrows. If the addition caused an additional ternary digit to be generated, subtract 0 from it, not 1.

```
         1  2  1  2  2  0
   +     1  1  1  1  1  1  (with carries)
-----------------------------
      1  0  1  0  1  0  1
  -   0  1  1  1  1  1  1  (without borrows)
----------------------------
      +  -  0  -  0  -  0
```

However, the following method may be easier on the abacus:
convert the number "in place", using the same convention for representing balanced ternary as before, with + represented by one earth bead next to the counter beam and all other beads away, - represented by the heaven bead next to the counter beam and all earth beads away, and 0 represented by having the heaven bead and only one earth bead next to the counter beam. Each 1 becomes a + and each 2

becomes a + - pair. As an example, convert the above ternary number, 121220 to balanced ternary:

We start at the left end:

```
      1  2  1  2  2  0
      +  2  1  2  2  0     convert 1 to a +
      +  -                 convert 2 to + -  pair and add the + to the previous +  (the sum = + -)
   +  -  -  1  2  2  0
   +  -  -  +  2  2  0     convert 1 to +
            +  -           convert 2 to + - and add the + to the previous +  (the sum = + -)
   +  -  0  -  -  2  0     the + carry from the + + sum combines with - to equal 0
               +  -        convert 2 to + - and add the + to the previous - to equal 0
   +  -  0  -  0  -  0     ternary 0 becomes balanced ternary 0, and we are finished
```

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